# MATHCOUNTS®

# Divisibility Rules

**Note:** *The terms in blue italics commonly appear in competition problems. Make sure Mathletes understand their meaning!*

**Coach instructions:** *These problems are from old countdown rounds. They are solvable in 45 seconds or less. Ask students to try to solve them all in 5 minutes or less. They might not be able to do this on the initial try. If students find them challenging, allow them to come back to these after watching the video to see if they can find the quick and elegant solutions!*

## Warm-Up!

*Try these problems before watching the lesson.*

1. Two single-digit positive ***integers*** have a product of 72. What is their sum?

   **We know 72 is a multiple of 9 because $7 + 2 = 9$. The other factor when 72 is divided by 9 is 8. So we have $8 + 9 = 17$.**

2. What is the sum of all two-digit multiples of three that have units digit 1?

   **The two-digit numbers with units digit 1 are 11, 21, 31, 41, 51, 61, 71, 81 and 91. We could test each one individually, but we know a number is divisible by 3 if the digits sum to a multiple of 3. This means the first number divisible by 3 is 21 because $2 + 1 = 3$. The other two numbers will be 51 and 81 because $5 + 1 = 6$ and $8 + 1 = 9$. The sum of these numbers is $21 + 51 + 81 = 153$.**

3. What is the remainder when 38 dozen is divided by 7?

   **One way to solve this is to multiply 38 by 12 then divide by 7:**
   **$38 \times 12 = 30 \times 12 + 8 \times 12 = 360 + 96 = 456$**
   **$456 \div 7 = 65$ R 1**

   **An arguably quicker way to solve this is to look at the remainders of 38 and 12 divided by 7 separately. 38 divided by 7 would be remainder 3 and 12 would be remainder 5. Multiplying $3 \times 5$ gets us 15 or $14 + 1$.**

   **To demonstrate why this works, consider the following:**
   **$38 \times 12 = (35 + 3)(7 + 5) = 35 \times 7 + 35 \times 5 + 3 \times 7 + 3 \times 5$**
   **As shown, when we multiply these binomials, we will get 4 terms of which 3 will be multiples of 7, so we only need to consider $3 \times 5 = 15$. Since $15 = 14 + 1$ we know the remainder will be 1.**

4. What is the ***least common multiple*** of 6, 10 and 14?

   **The prime factorizations of these three numbers are:**
   **$6 = 2 \times 3$**
   **$10 = 2 \times 5$**
   **$14 = 2 \times 7$**
   **So the least common multiple will be $2 \times 3 \times 5 \times 7 = 210$.**

5. Kris multiplies the first six positive ***prime*** numbers together. How many zeros follow the last non-zero digit of the product?

**To solve this, you could multiply out the product:**
**$2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30030$**
**Here we can see that after the last non-zero digit of the product there is 1 zero.**

**However, you can determine this by simply looking at the factors we are multiplying. Every zero after the last non-zero digit in a number represents divisibility by a factor of 10. If there is 1 zero it's divisible by 10, 2 zeros it's divisible by 100, 3 zeros it's divisible by 1000, and so on... So we really just need to see how many 10's can be factored out of the expression. Since there is one 2 and one 5 we know there will be 1 zero.**

# The Problems

*Take a look at the following problems and follow along as they are explained in the video.*

***Coach instructions:** After students try the warm-up problems, play the video and have them follow along with the solutions. After watching the video, they may want to go back and try to come up with faster ways to solve the warm-up problems before moving on to the final problem set.*

6. Using each of the four digits 2, 4, 6 and 8 exactly once, how many four-digit multiples of 4 can be formed?

**Solution in video. Answer: 12.**

7. How many zeros are there after the last nonzero digit of 125!?

**Solution in video. Answer: 31.**

8. What is the greatest number that evenly divides the sum of any six ***consecutive*** whole numbers?

**Solution in video. Answer: 3.**

# Piece It Together

*Use the skills you practiced in the warm-up and strategies from the video to solve the following problems.*

***Coach instructions:** After watching the video, give students 15-20 minutes or so to try the next seven problems.*

9. Ayasha, Beshkno, and Chenoa were all born after 2000. Each of them was born in a year after 2000 that is divisible by exactly one of the prime numbers 2, 3 or 5. Each of these primes is a divisor of one of the birth years. What is the least possible sum of their birth years?

**The first number after 2000 that is divisible by 2 is 2002—the next even number. We can quickly see that it isn't divisible by either 3 or 5 because the sum of the digits is 2 + 0 + 0 + 2 = 4 and the last digit isn't a 0 or 5.**

**The first number after 2000 that is divisible by 3 is 2001 because 2 + 0 + 0 + 1 = 3. We**

**know this number isn't divisible by 2 because it is not even and not divisible by 5 because it does not end in a 0 or 5.**

**Finally, the first number after 2000 that is divisible by 5 is 2005—the next number to end in a 0 or 5. It is not even and not divisible by 3 because 2 + 0 + 0 + 5 = 7.**

**So the least possible sum of their birth years is 2002 + 2001 + 2005 = 6008.**

10. The product of three ***consecutive integers*** is 157,410. What is their sum?

**Let's use our divisibility knowledge to factor 157,410:**

| | |
|---|---|
| **157,410 = 10 × 15,741** | **the number ends in 0, 10 is a factor** |
| **15,741 = 9 × 1749** | **the digit sum is 1 + 5 + 7 + 4 + 1 = 18, 9 is a factor** |
| **1749 = 11 × 159** | **the alternating ± digits sum is 1 − 7 + 4 − 9 = −11, 11 is a factor** |
| **159 = 3 × 53** | **the digit sum is 1 + 5 + 9 = 15, 3 is a factor** |

**The prime factorization of 157,410 is $2 \cdot 3^3 \cdot 5 \cdot 11 \cdot 53$. We know 53 is one of the consecutive integers and can then see that $5 \cdot 11 = 55$ and $2 \cdot 3^3 = 54$ are the other two. Their sum is 53 + 54 + 55 = 162.**

11. What is the least positive integer *n* for which 165 × 513 + 10*n* is a multiple of 9?

**You might be inclined to multiply out the first term and try to write an expression for the sum of the digits of the entire expression, and then do test cases for the integer *n*, but let's look at the two terms of the expression separately first.**

**The first term is 165 × 513. Notice that 513 is divisible by 9 because the digit sum is 5 + 1 + 3 = 9. This means the entire term will be divisible by 9 since it is a factor of the expression. We do not have to multiply out and find the value.**

**So really, we just need to find what value of *n* makes 10*n* a multiple of 9. The smallest positive integer value that makes this true is 9. Since 9 is a factor of both terms of the expression, we know the whole expression is factorable by 9.**

12. What is the least positive integer greater than 20 that has exactly three positive factors?

**Before you start listing out the factors of integers greater than 20, think about what is special about a number with three positive factors. When we think of factors we usually think of them as coming in factor pairs. So for 20, for example, we have the factor pairs 1 × 20, 2 × 10 and 4 × 5, which means 20 has six positive factors.**

**In order for a number to have an odd number of positive factors, it needs to be a perfect square. The smallest perfect square greater than 20 is 25. The three positive factors of 25 are 1, 5 and 25.**

13. What is the least positive integer $n$ such that the value of $2014!/n!$ does not have a units digit of zero?

**The expression $2014!/n!$ will have a units digit of zero if there are any multiples of 10. Looking at 2014!, we can write it out as $2014! = 2014 \cdot 2013 \cdot 2011 \cdot 2012 \cdot 2010 \cdots 3 \cdot 2 \cdot 1$.**

**The first occurrence of a multiple of 10 is 2010. So if we choose $n$ to be 2010, we will guarantee to divide out all integers with 10 as a factor.**

14. How many positive 3-digit integers are ***palindromes*** and multiples of 11?

**The rule for determining a multiple of 11 is that the alternating addition subtraction digit sum is a multiple of 11. A 3-digit palindrome will be of the form ABA where letters represent digits. We have two cases we can test using this rule for 11's:**
**(1) $A - B + A = 0$ or $2A = B$**
**(2) $A - B + A = 11$ or $2A - 11 = B$.**

**Let's start by looking at A = 1 and moving through all nine digit options. If A = 1, then case (1) gives us the palindrome 121. Similarly, case (1) gives us 242, 363 and 484. If A = 5, neither case can be satisfied. If A = 6, then case (2) gives us 616. Similarly, case (2) gives us 737, 858 and 979.**

**So there are 8 3-digit integer palindromes that are multiples of 11: 121, 242, 363, 484, 616, 737, 858 and 979.**

15. The six-digit number 357,abc has six distinct digits and is divisible by each of 3, 5 and 7. What is the smallest possible value of $a + b + c$?

**The first part of the six-digit number is 357. We can see this will be divisible by 7 since 35 and 7 are both divisible by 7 (note: also by the rule for $7 \rightarrow 35 - 14 = 21$). This number is also divisible by 3 because $3 + 5 + 7 = 15$. Since divisibility by 5 is determined by the final digit, we can use this information to focus on just the final three digits, abc.**

**Since we want to minimize $a + b + c$, let's first test $c = 0$ to give us divisibility by 5. The other two digits must be distinct and if we choose 2 and 1, we can construct the number 210 which is divisible by 3, 5 and 7. The larger number 357,210 is also divisible by 3, 5 and 7. This makes the smallest possible value of $a + b + c = 2 + 1 + 0 = 3$.**

# Optional Extension

**Coach instructions:** *If students want to explore these rules further, they can do this extension. This may be time consuming and some rules are easier to prove than others. You might want to assign students one or two rules to focus on based on their ability.*

*To extend your understanding and have a little fun with math, try the following activities.*

So you have learned and applied some divisibility rules for various integers, but why do these rules work? Work your way through the divisibility rules and try to explain or prove them. Some might be easier for you than others. Start with 2 and work your way up or bounce around to the ones that make the most sense to you. Try to explain as many as you can!

**Divisibility Rules**

| # | Rule |
|---|---|
| 2 | The number has a units digit 0, 2, 4, 6 or 8. |
| 3 | The sum of the digits is a multiple of 3. |
| 4 | The number formed by the last two digits is divisible by 4. |
| 5 | The number has a units digit 0 or 5. |
| 6 | The number is even, and its digits sum to a multiple of 3. |
| 7 | The result of subtracting twice the units digit from the number formed by the remaining digits is divisible by 7. |
| 8 | The number formed by the last three digits is divisible by 8. |
| 9 | The sum of the digits is a multiple of 9. |
| 10 | The number has a units digit of 0. |
| 11 | The alternating addition and subtraction of the digits is a number divisible by 11. |

## Divisibility by 2

*Hint:* Prove 5's rule the same way. 5, 10, 15, 20, 25, 30, …

**If we start with 2 and continuously add/count by 2, we will notice a pattern:**
**2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, …**
**The units digit repeats 2, 4, 6, 8, 0, 2, 4, 6, 8, 0….**
**Thus, if a number ends in one of these digits, it will be divisible by 2.**

## Divisibility by 3

*Hint:* Substitute 9 for 3 in this proof to prove 9's rule

**10 divided by 3 has remainder 1 since 10 = 9 + 1**
**100 divided by 3 has remainder 1 since 100 = 99 + 1**
**1000 divided by 3 has remainder 1 since 1000 = 999 + 1 and so on…**

**Let's consider an unknown three-digit integer ABC.**
**We can rearrange this number as follows:**

$$\begin{aligned} ABC &= 100A + 10B + C \\ &= (99 + 1)A + (9 + 1)B + C \\ &= 99A + 9B + A + B + C \end{aligned}$$

In this new arrangement, the first two terms are both divisible by 3. If the remainder of the terms are also divisible by 3, then we know the number ABC is also divisible by 3. Notice that what we have left is the sum of the digits of our original number hence the divisibility rule.

### Divisibility by 4

Since $100 = 4 \times 25$, any multiple of 100 is also divisible by 4. This is why we only have to consider the number formed by the final two digits to determine divisibility by 4.

*Hint:* $1000 = 8(125)$ Modify this proof for 8's rule.

Ex:

$$
\begin{aligned}
79556 &= 79500 + 56 \\
&= 795(100) + 56 \\
&= 795(4)(25) + 4(14)
\end{aligned}
$$

### Divisibility by 6

The rule for divisibility by 6 is the combination of the rules for 2 and 3. This is because all numbers divisible by $6 = 2 \times 3$ are divisible by both 2 and 3. So, if we check for divisibility by both 2 and 3, we guarantee it is divisible by 6.

### Divisibility by 7

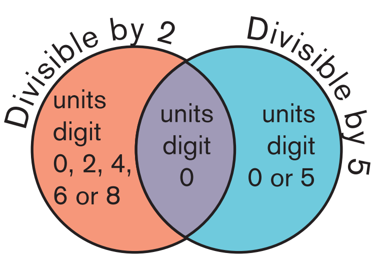
This one is a little tricky... Let's start with a number AB. Here B represents the units digit and A represents the rest of the number—it could be any number of digits long.

The proof will go as follows:

| | |
|---|---|
| $10A + B = 7K$ | Assume our number is equivalent to some multiple of 7 |
| $10A - 20B = 7K - 21B$ | Subtract 21B from both sides |
| $10(A - 2B) = 7(K - 3B)$ | Factor both sides |

Since A, B and K are all integers and 7 is prime, we know $A - 2B$ is divisible by 7 (b/c 10 is not). Thus, if our number is a multiple of 7, $A - 2B$ is also a multiple of 7.

### Divisibility by 10




One way to think of it is that the rule for divisibility by 10 is the overlap or intersection of the rules for divisibility by 2 and 5. This is because all numbers divisible by 10 are divisible by both 2 and 5. Visually, a Venn Diagram could be helpful for understanding this.

*Note: This method can be used to determine divisibility by other composite numbers.*

Alternatively, you could approach this the same way as divisibility by 2 or 5. If we count by ten, we will notice that all the numbers end in zero: 10, 20, 30, 40, 50, …

## Divisibility by 11

The rule for divisibility of 11 tells us to alternate adding and subtracting the digits. If the result is divisible by 11, the original number is divisible by 11. Similar to our proofs for 3 and 9, let's look at 11's relationship to powers of 10:

$10 = 11 - 1$

$$\begin{aligned} 100 &= 10 \cdot 10 \\ &= 10(11 - 1) \\ &= 10 \cdot 11 - 10 \\ &= 10 \cdot 11 - (11 - 1) \\ &= 9 \cdot 11 + 1 \end{aligned}$$

$$\begin{aligned} 1000 &= 100 \cdot 10 \\ &= 100(11 - 1) \\ &= 100 \cdot 11 - 100 \\ &= 100 \cdot 11 - (9 \cdot 11 + 1) \\ &= 91 \cdot 11 - 1 \end{aligned}$$

We see that 10 is one less than a multiple of 11, 100 is one more, 1000 is one less, and so on. We could have established this by just stating $10 = 11 - 1$, $100 = 99 + 1$ and $1000 = 1001 - 1$, but the intension here was to show there was a pattern that will continue through all powers of 10.

Now let's use this to prove why the alternating addition and subtraction of digits works. If we look at a four-digit integer ABCD, we can rearrange it as follows:

$$\begin{aligned} ABCD &= 1000A + 100B + 10C + D \\ &= (91 \cdot 11 - 1)A + (9 \cdot 11 + 1)B + (11 - 1)C + D \\ &= 91 \cdot 11A + 9 \cdot 11B + 11C - A + B - C + D \end{aligned}$$

So, we can see that divisibility by 11 is truly determined by the alternating sum of digits.