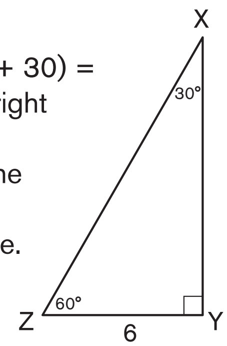
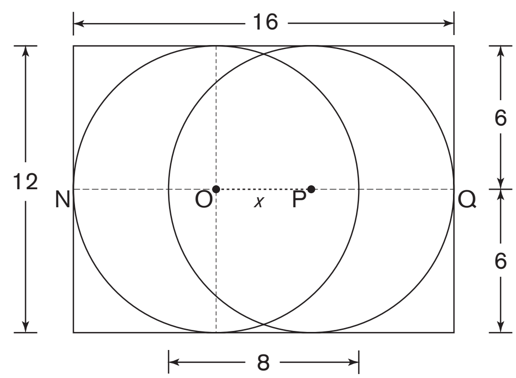
MATHCOUNTS® Minis

March 2015 Activity Solutions

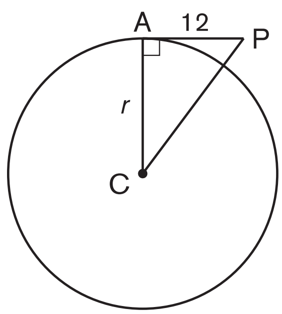
Warm-Up!



1. Since the $m\angle X = 30°$ and $m\angle Y = 90°$, it follows that the $m\angle Z = 180 - (90 + 30) = 180 - 120 = 60°$. Therefore, $\triangle XYZ$ is a 30-60-90 right triangle. In a 30-60-90 right triangle, the length of the longer leg, which is opposite the 60° angle, is $\sqrt{3}$ times the length of the shorter leg, which is opposite the 30° angle, and the length of the hypotenuse is twice that of the shorter leg. As the figure shows, in triangle XYZ, side YZ is the shorter leg, side XY is the longer leg and side XZ is the hypotenuse. We are told YZ = 6 units, so it follows that XY = $\mathbf{6\sqrt{3}}$ units and XZ = **12** units.



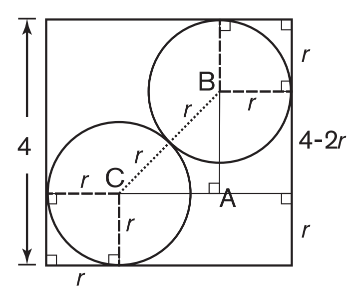
2. Based on the information provided in the problem, we've indicated many measurements of the figure here. Since the width of the rectangle is also the diameter of each circle, we know that the radius of each circle is 6 units. Since NO + OP + PQ = 16, with segments NO and PQ being radii, we can write the equation $6 + x + 6 = 16$, when OP = x. Simplifying leads to $6 + x + 6 = 16 \rightarrow x + 12 = 16 \rightarrow x = \mathbf{4}$ units.



3. Since segment AP is tangent to the circle at A, segment PA will be perpendicular to radius AC. Because the area of the circle is 256π units^2, we can write the following equation and solve for r: $256\pi = \pi r^2 \rightarrow 256 = r^2 \rightarrow r = 16$ units. Using the Pythagorean Theorem with right triangle APC, we now can write the following equation and solve for PC: $(PC)^2 = 12^2 + 16^2 \rightarrow (PC)^2 = 144 + 256 \rightarrow (PC)^2 = 400 \rightarrow PC = \mathbf{20}$ units.

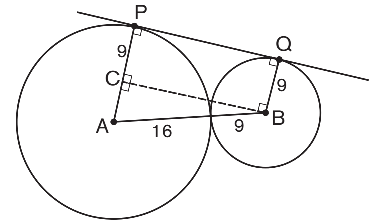
The Problems are solved in the MATHCOUNTS® Mini video.

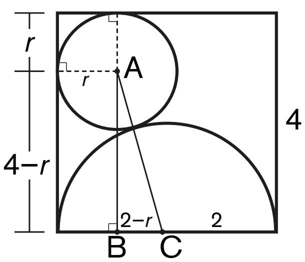
Follow-up Problems



4. In the figure shown here, we have indicated the different radii and right angles that are known from the information given. Additionally, since each side of the square is 4 units, we can see how the right side is divided into segments of lengths r units, r units and $4 - 2r$ units. Thus, $AB = 4 - 2r$. Similarly, $AC = 4 - 2r$. Seeing that $BC = 2r$ and using the Pythagorean Theorem with right triangle ABC, we can write the following equation and solve for r: $(2r)^2 = (4 - 2r)^2 + (4 - 2r)^2 \rightarrow 4r^2 = 16 - 16r + 4r^2 + 16 - 16r + 4r^2 \rightarrow 0 = 4r^2 - 32r + 32 \rightarrow 0 = r^2 - 8r + 8$. Using the Quadratic Formula with $a = 1$, $b = -8$ and $c = 8$, we get $r = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(8)}}{2(1)} = \dfrac{8 \pm \sqrt{64 - 32}}{2} = \dfrac{8 \pm \sqrt{32}}{2} = \dfrac{8 \pm 4\sqrt{2}}{2} = 4 \pm 2\sqrt{2}$. Since $4 + 2\sqrt{2}$ is too large (it's greater than the side of the square), the radius is $\mathbf{4 - 2\sqrt{2}}$ units.

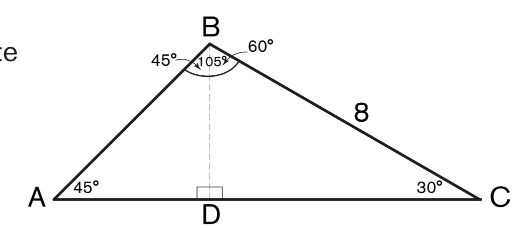
5. In the figure shown here, we have added the segment from B that is perpendicular to radius AP. This segment completes rectangle BCPQ, and now BQ = PC, so PC = 9 units. Radius AP is 16 units, so AC = 16 − 9 = 7 units. When we connect the two centers of the externally tangent circles, we get AB = 16 + 9 = 25 units. Now, using the Pythagorean Theorem with right triangle ABC, we have $25^2 = 7^2 + (BC)^2 \rightarrow 625 = 49 + (BC)^2 \rightarrow (BC)^2 = 576 \rightarrow BC = 24$ units. Because of rectangle BCPQ, we now know PQ = **24** units, too.





6. Let's start by drawing a segment from center point A perpendicular to the side of the square (also the diameter of the semicircle) at point B, and drawing another segment from A to C, the midpoint of that same side, as shown. Let r represent the radius of the circle. Then AB = $4 - r$, BC = $2 - r$ and AC = $r + 2$. Using the Pythagorean Theorem with right triangle ABC, we have $(4 - r)^2 + (2 - r)^2 = (r + 2)^2 \rightarrow (16 - 8r + r^2) + (4 - 4r + r^2) = r^2 + 4r + 4 \rightarrow 2r^2 - 12r + 20 = r^2 + 4r + 4 \rightarrow r^2 - 16r + 16 = 0$. Since this quadratic cannot be factored, we'll use the quadratic formula, $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve for r. Substituting $a = 1$, $b = -16$, $c = 16$ into the formula, we have $r = \dfrac{16 \pm \sqrt{16^2 - 4 \times 1 \times 16}}{2 \times 1}$. Solving for r, we get $r = \dfrac{16 \pm \sqrt{256 - 64}}{2} = \dfrac{16 \pm \sqrt{192}}{2} = \dfrac{16 \pm 8\sqrt{3}}{2} = 8 \pm 4\sqrt{3}$. Since $8 - 4\sqrt{3} > 4$, this root will not work. Therefore, $r = \mathbf{8 - 4\sqrt{3}}$ units.

7. Consider △ABC, in which, $m\angle A = 45°$, and the side opposite ∠A, side BC, has length 8 units. If we draw a line from vertex B to a point D on side AC so that segment BD is perpendicular to side AC, △ABC is divided into two smaller triangles. As the figure shows, △ABD is a 45-45-90 right triangle, and △BCD is a 30-60-90 right triangle. We are asked to determine the sum of the two missing side lengths of △ABC, AB + AC. Since △BCD is a 30-60-90 right triangle with hypotenuse BC of length 8 units, it follows that the shorter leg, side BD has length 4 units and the longer leg, side DC has length $4\sqrt{3}$ units. Now, since △ABD is a 45-45-90 right triangle with leg BD of length 4 units, it follows that leg AD also has length 4 units and hypotenuse AB has length $4\sqrt{2}$ units. We now know the missing side lengths for △ABC are AB = $4\sqrt{2}$ units and AC = $4 + 4\sqrt{3}$ units, so AB + AC = $\mathbf{4\sqrt{2} + 4 + 4\sqrt{3}}$ units.



8. Let's extend segments AD and BC until they intersect at point E, as shown. Notice that $m\angle EBA = 180 - 120 = 60°$, and $m\angle BAE = 180 - 90 = 90°$. That means the $m\angle E = 30°$, and △ABE is a 30-60-90 right triangle. We know that AB = 3 units, so using the properties of 30-60-90 right triangles, EB = 6 units. Now consider right triangle CDE with $m\angle C = 90°$ and $m\angle E = 30°$. It follows that $m\angle D = 60°$ making △CDE a 30-60-90 right triangle. The length of the longer leg is EC = EB + BC = 6 + 4 = 10 units. Segment CD is the shorter leg of △CDE. Therefore, according to the properties of 30-60-90 right triangles, we have CD = $\dfrac{EC}{\sqrt{3}} = \dfrac{10}{\sqrt{3}} = \mathbf{\dfrac{10\sqrt{3}}{3}}$ units.