MATHCOUNTS® Minis

April 2015 Activity Solutions

Warm-Up!

1. We can solve the quadratic equation $x^2 - x - 6 = 0$ by factoring to get: $(x - 3)(x + 2) = 0$, so $x - 3 = 0$ and $x = 3$, or $x + 2 = 0$ and $x = -2$. The product of the two solutions is $3 \times (-2) = \mathbf{-6}$.

2. From the information provided, we have the following equation: $M(M - 6) = -5$, which can be rewritten as the quadratic equation $M^2 - 6M + 5 = 0$. Factoring, we get $(M - 5)(M - 1) = 0$. So $M - 5 = 0$ and $M = 5$, or $M - 1 = 0$ and $M = 1$. The sum of the possible values of M is $5 + 1 = \mathbf{6}$.

3. Since $x = 3$ and $x = -4/3$ are the solutions to $kx^2 - 5x - 12 = 0$, we know $x - 3 = 0$ or $3x = -4$ $\rightarrow 3x + 4 = 0$. The quadratic, then, can be written as the product of the two binomials $x - 3$ and $3x + 4$ to get $(x - 3)(3x + 4) = 3x^2 + 4x - 9x - 12 = 3x^2 - 5x - 12$. We now can see that $k = \mathbf{3}$.

4. We can solve the cubic equation $x^3 + 3x^2 - 10x = 0$ by factoring to get: $(x)(x + 5)(x - 2) = 0$, so $x = 0$, or $x + 5 = 0$ and $x = -5$, or $x - 2 = 0$ and $x = 2$. The mean of these three solutions, then, is $(-5 + 0 + 2)/3 = -3/3 = \mathbf{-1}$.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. We are asked to find the value of $(a - 1)(b - 1)$. When we multiply using the distributive property, we get $ab - (a + b) + 1$. Now since a and b are solutions to the quadratic equation $x^2 - 5x + 9 = 0$, we can write $(x - a)(x - b) = 0 \rightarrow x^2 - (a + b)x + ab = 0$. That means $a + b = 5$ and $ab = 9$. Thus, $(a - 1)(b - 1) = ab - (a + b) + 1 = 9 - 5 + 1 = \mathbf{5}$.

6. Suppose m and n are the roots of the quadratic equation $x^2 - 63x + k = 0$. We know that $m + n = 63$ and $mn = k$. We are told that both roots are prime numbers. The smallest prime is 2, and $63 - 2 = 61$. Since 2 and 61 both are prime, we have the possible roots $m = 2$ and $n = 61$. In this case, $k = mn = 2 \times 61 = 122$. Since all prime numbers except 2 are odd, and the sum of any two odd primes will result in an even number, there is no other pair of primes that can have a sum of 63. Therefore, 122 is the the **1**, and only, possible value of k.

7. Let's begin by manipulating the quadtratic equation $(2003/2004)x + 1 + (1/x) = 0$ to get a leading coefficient of 1. We do so by multiplying both sides of the equation by 2004/2003 and get $x + 2004/2003 + (2004/2003)(1/x) = 0$. Next, we want there to be no fractions with variables in the denominator, so we can multiply both sides of the equation by x to get $x^2 + (2004/2003)x + 2004/2003 = 0$. If p and q are roots of this equation, we are asked to find the sum of their reciprocals, $1/p + 1/q = (p + q)/pq$. Since p and q are roots, it follows that $(x - p)(x - q) = 0 \rightarrow x^2 - (p + q)x + pq = 0$. That means $p + q = -2004/2003$ and $pq = 2004/2003$. Substituting these values, we see that the sum of the reciprocals of the roots is $(p + q)/pq = (-2004/2003)/(2004/2003) = (-2004/2003)(2003/2004) = \mathbf{-1}$.

8. If a and b are the roots of $x^2 + px + m = 0$, we know that $m = ab$ and $p = -(a + b)$. We are told that the roots of $x^2 + mx + n = 0$ are $2a$ and $2b$, so we know that $n = 4ab$ and $m = -2(a + b)$. Since $m = ab$ and $n = 4ab$, it follows that $n = 4m$. And since $m = -2(a + b)$ and $p = -(a + b)$, it follows that $p = m/2$. Therefore, $n/p = 4m/(m/2) = 4m \times (2/m) = \mathbf{8}$.

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