Warm-Up!

1. We can solve the quadratic equation \( x^2 - x - 6 = 0 \) by factoring to get: \((x - 3)(x + 2) = 0\), so \(x - 3 = 0\) and \(x = 3\), or \(x + 2 = 0\) and \(x = -2\). The product of the two solutions is \(3 \times (-2) = -6\).

2. From the information provided, we have the following equation: \(M(M - 6) = -5\), which can be rewritten as the quadratic equation \(M^2 - 6M + 5 = 0\). Factoring, we get \((M - 5)(M - 1) = 0\). So \(M - 5 = 0\) and \(M = 5\), or \(M - 1 = 0\) and \(M = 1\). The sum of the possible values of \(M\) is \(5 + 1 = 6\).

3. Since \(x = 3\) and \(x = -4/3\) are the solutions to \(kx^2 - 5x - 12 = 0\), we know \(x - 3 = 0\) or \(3x = -4\), so \(3x + 4 = 0\). The quadratic, then, can be written as the product of the two binomials \(x - 3\) and \(3x + 4\) to get \((x - 3)(3x + 4) = 3x^2 + 4x - 9x - 12 = 3x^2 - 5x - 12\). We can now see that \(k = 3\).

4. We can solve the cubic equation \(x^3 + 3x^2 - 10x = 0\) by factoring to get: \((x)(x + 5)(x - 2) = 0\), so \(x = 0\), or \(x + 5 = 0\) and \(x = -5\), or \(x - 2 = 0\) and \(x = 2\). The mean of these three solutions, then, is \((-5 + 0 + 2)/3 = -3/3 = -1\).

The Problems are solved in the MATHCOUNTS video.

Follow-up Problems

5. We are asked to find the value of \((a - 1)(b - 1)\). When we multiply using the distributive property, we get \(ab - (a + b) + 1\). Now since \(a\) and \(b\) are solutions to the quadratic equation \(x^2 - 5x + 9 = 0\), we can write \((x - a)(x - b) = 0 \rightarrow x^2 - (a + b)x + ab = 0\). That means \(a + b = 5\) and \(ab = 9\). Thus, \((a - 1)(b - 1) = ab - (a + b) + 1 = 9 - 5 + 1 = 5\).

6. Suppose \(m\) and \(n\) are the roots of the quadratic equation \(x^2 - 63x + k = 0\). We know that \(m + n = 63\) and \(mn = k\). We are told that both roots are prime numbers. The smallest prime is 2, and 63 - 2 = 61. Since 2 and 61 both are prime, we have the possible roots \(m = 2\) and \(n = 61\). In this case, \(k = mn = 2 \times 61 = 122\). Since all prime numbers except 2 are odd, and the sum of any two odd primes will result in an even number, there is no other pair of primes that can have a sum of 63. Therefore, 122 is the one, and only, possible value of \(k\).

7. Let's begin by manipulating the quadratic equation \((2003/2004)x + 1 + (1/x) = 0\) to get a leading coefficient of 1. We do so by multiplying both sides of the equation by \(2004/2003\) and get \(x + 2004/2003 + (2004/2003)(1/x) = 0\). Next, we want there to be no fractions with variables in the denominator, so we can multiply both sides of the equation by \(x\) to get \(x^2 + (2004/2003)x + 2004/2003 = 0\). If \(p\) and \(q\) are roots of this equation, we are asked to find the sum of their reciprocals, \(1/p + 1/q = (p + q)/pq\). Since \(p\) and \(q\) are roots, it follows that \((x - p)(x - q) = 0 \rightarrow x^2 - (p + q)x + pq = 0\). That means \(p + q = -2004/2003\) and \(pq = 2004/2003\). Substituting these values, we see that the sum of the reciprocals of the roots is \((p + q)/pq = (-2004/2003)/(2004/2003) = -1\).

8. If \(a\) and \(b\) are the roots of \(x^2 + px + m = 0\), we know that \(m = ab\) and \(p = -(a + b)\). We are told that the roots of \(x^2 + mx + n = 0\) are \(2a\) and \(2b\), so we know that \(n = 4ab\) and \(m = -2(a + b)\). Since \(m = ab\) and \(n = 4ab\), it follows that \(n = 4m\). And since \(m = -2(a + b)\) and \(p = -(a + b)\), it follows that \(p = m/2\). Therefore, \(n/p = 4m/(m/2) = 4m \times (2/m) = 8\).