MATHCOUNTS® Minis

October 2017 Activity Solutions

Warm-Up!

1. The first term of the sequence is 2, and each term that follows is 3 times the previous term. To get the second term, we multiply the first term by 3 once. To get the third term, we multiply the first term by 3 twice. It follows that to get the seventh term, we need to multiply the first term by 3 six times. In other words, we need to multiply the first term by 3^6. Therefore, the seventh term of the geometric sequence is $2 \times 3^6 = 2 \times 729 =$ **1458**.

2. Figure 1 has 1 dot. Figure 2 has 3 dots, which is 2 more that the previous figure. Figure 3 has 6 dots, which is 3 more than the previous figure. Finally, Figure 4 has 10 dots, which is 4 more than the previous figure. Notice the pattern shown in the table below.

FIGURE	DOTS
1	1
2	$1 + 2 = 3$
3	$1 + 2 + 3 = 6$
4	$1 + 2 + 3 + 4 = 10$
⋮	⋮
n	$1 + 2 + 3 + 4 + \cdots + n$

Therefore, Figure 10 has $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =$ **55** dots. The numbers of dots in each figure form a sequence of numbers commonly referred to as the Triangular Numbers. There is a formula to determine the sum of the first n positive integers. It is $1 + 2 + 3 + 4 + \cdots + n = n(n + 1)/2$. So, in this case, the number of dots in Figure 10, which represents the tenth Triangular Number, is $10(11)/2 = 110/2 =$ **55** dots.

3. To evaluate $(4 - 3) + (5 - 4) + (6 - 5) + (7 - 6) + \cdots + (2010 - 2009)$, we need to first evaluate the expression in each set of parentheses. The value of the expression in each of the $2010 - 4 + 1 = 2007$ sets of parentheses is 1. So, the sum is $1 \times 2007 =$ **2007**.

4. To evaluate $5/3 \times 6/4 \times 7/5 \times 8/6 \times \cdots \times 120/118$, we can start by cancelling common factors in the numerator and denominator. Doing so, results in the much simpler expression $(119 \times 120)/(3 \times 4) = 119 \times 10 =$ **1190**.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. The expression $3 + 6 - 9 + 12 + 15 - 18 + 21 + 24 - 27 + \cdots + 84 + 87 - 90$ is the sum of the multiples of 3 from 3 to 90, inclusive, that are not multiples of 9, minus the multiples of 9 from 9 to 90, inclusive. Evaluating, we get $(3 + 6 + 9 + \cdots + 90) - 2 \times (9 + 18 + 27 + \cdots + 90) =$ $[(3 + 90) \times (90 \div 3)] \div 2 - [2 \times [(9 + 90) \times (90 \div 9)] \div 2] = (93)(15) - (99)(10) =$ **405**.

6. When the dots are connected, triangles are created, as shown. Instead of looking at the line segments, let's look at the number of shaded triangles in each figure.

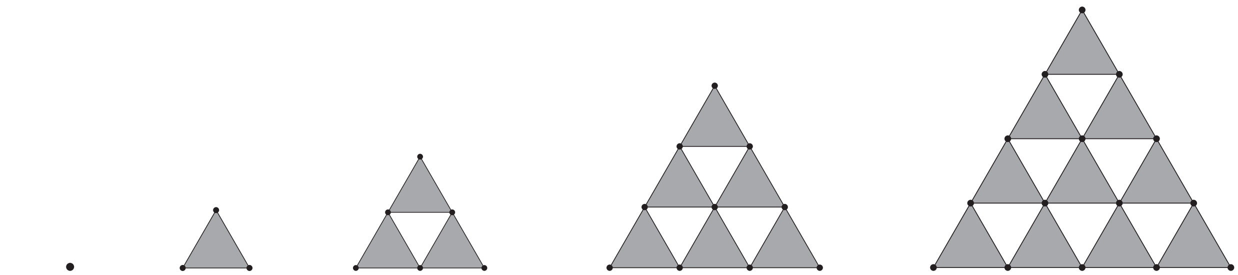

Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

Figure 1 has no triangles. Figures 2, 3, 4 and 5 each have 1, 3, 6 and 10 shaded triangles, respectively. The sequence representing the number of shaded triangles in each figure is 0, 1, 3, 6, 10, ... Recall, that the sequence representing the number of dots in each figure is 1, 3, 6, 10, 15, ... Notice that these two sequences are the same, except the terms are one-off because the first term of the sequence representing the shaded triangles is 0. It follows, then, that in Figure 10 there are $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ shaded triangles. Each shaded triangle has a perimeter of 3. That means the length of all the segments in Figure 10 is $45 \times 3 =$ **135** units.

7. Let's count the squares of the stages shown to see what pattern is emerging. Stage 1 starts with 1 square. From stage 1 to stage 2, 4 squares are added. From stage 2 to stage 3, 8 squares are added. From stage 3 to stage 4, 12 squares are added. The number of squares is increasing following the arithmetic sequence $1 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + \ldots. + 4 \cdot (n - 1)$, where n is the stage number. At stage 10, there will be $1 + 4 \cdot 1 + 4 \cdot 2 + \ldots + 4 \cdot 9 = 1 + 4(1 + 2 + \ldots + 9) = 1 + 4(45) = 1 + 180 =$ **181**.

8. To determine the number of distinct arithmetic sequences having 1 as the first term with 91 in the sequence, we first need to determine how far it is from 1 to 91. Since $91 - 1 = 90$ and the sequence it to contain only integers, we need to determine the number of ways we can divide 90 into integral parts. In other words, what are the factors of 90? The factors of 90 are 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45 and 90. These are the twelve possible values for the common difference, d. That means there are **12** distinct arithmetic sequences that meet the given conditions.