MATHCOUNTS® Minis November 2014 Activity Solutions

Warm-Up!

1. We are asked to determine the total number of sequences of the five coin flips that result in 4 heads (*H*) and 1 tail (*T*). If we think of the results filling five spaces, one such sequence would be:

<u>*H*</u> <u>*H*</u> <u>*H*</u> <u>*H*</u> <u>*T*</u>

The only thing that differentiates sequences of the 4 *H*s and 1 *T* is the location of the *T*. Since the *T* can be in one of five locations, there are **5** different sequences: *HHHHT*, *HHHTH*, *HHTHH*, *HTHHH* and *THHHH*.

2. We are asked to determine the total number of sequences of the five coin flips that result in 3 heads and 2 tails. As with the previous problem we can consider the different locations of the two *T*s to determine the different sequences. Each of the two *T*s will fill one of the five spaces, so that's 5 choose 2. Therefore, there are ${}_5C_2$ = 5!/(3! 2!) = (5 × 4)/2 = 20/2 = 10 ways to place the 2 *T*s, so there are **10** different sequences.

3. Let's first consider the probability that when flipping a fair coin 3 times the total number of heads (*H*) is greater than the total number of tails (*T*). The possible outcomes when flipping a coin 3 times are *HHH*, *HHT*, *HTH*, *THH*, *TTT*, *TTH*, *THT*, *HTT*. Notice that in half of these outcomes the total number of heads is greater than the total number of tails. This will be the case each time the coin is flipped an odd number of times. Therefore, the probability of the total number of heads being greater than the total number of tails when flipping a fair coin 37 times is **1/2**.

4. If the probability of this coin coming up heads is twice as likely as it coming up tails, we have that $P(H) = 2(P(T))$. Since $P(H) + P(T) = 1$, we have $2(P(T)) + P(T) = 1 \rightarrow 3(P(T)) = 1$, and $P(T) = 1/3$. That means that $P(H) = 2/3$. There are four ways for this coin to come up with 3 *H*s and 1 *T* in four flips: *HHHT*, *HHTH*, *HTHH* and *THHH*. The probability of each of these sequences occurring is the same, and each one is $(2/3)^3 \times (1/3) = 8/81$. Therefore, the probability of getting 3 *H*s and 1 *T* when this particular coin is flipped four times is 4 × (8/81) = **32/81**.

The Problem is solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Let $p = P(R)$, the probability of reaching into the bag and pulling out one red (*R*) ball. It then follows that $1 - p = P(\text{white})$, the probability of reaching in and pulling out a white (*W*) ball. We now need to write an expression for the probability of pulling out 2 *R*s and 3 *W*s, with replacement. The probability of pulling out five balls in the sequence *WWRRR* is $(1 - p)^2 \times p^3$. There are ${}_5C_2$ = 5!/(3! 2!) = 20/2 = 10 ways to arrange, or pull 2 *W*s and 3 *R*s, which gives us a probability of $10 \times (1 - p)^2 \times p^3$. Similarly, we need to find an expression for the probability of pulling 4 *W*s and 1 *R*. Again, we'll consider the probability of pulling out five balls in the sequence *WWWWR*, which is $(1 - p)^4 \times p$. Since there are ${}_5C_1$ = 5!/(4! 1!) = 5/1 = 5 ways to arrange, or pull 4 *W*s and 1 *R*, the probability is $5 \times (1 - p)^4 \times p$. We are told that the probability of getting 3 *R*s is 32 times the probability of getting just one *R*. We have the following equation: $10 \times (1 - p)^2 \times p^3 = 32 \times 5 \times (1 - p)^4 \times p$. Dividing each side by the quantity $10 \times p \times (1 - p)^2$, we get $p^2 = 16 \times (1 - p)^2$. Both p and $(1 - p)$ are positive, so we take the square root of each side to see that, originally, $p = 4(1 - p) \rightarrow p = 4 - 4p \rightarrow 5p = 4 \rightarrow p = 4/5$ = **80**% of the balls in the bag were red.

6. We're asked to determine the probability that two vertices chosen at random from the six vertices of a regular octahedron are the endpoints of an edge. This probability is the number of ways to successfully choose a pair of vertices that are an endpoint of an edge over the total number of ways to choose a pair of vertices. For each edge there is exactly one pair of vertices that are the endpoints of that edge. So, essentially, the number of edges is the number of suceesses. There are four edges between the four vertices around the middle, four edges from each of those vertices to the top vertex and four edges from each of the vertices to the bottom vertex. That's total of $4 \times 3 = 12$ successes. When selecting a pair of vertices, there are six ways to choose the first vertex and five ways to chose the second vertex. But this counts each of our pairs twice, so there are actually $(6 \times 5) \div 2 = 30 \div 2 = 15$ ways to select a pair of vertices. Therefore, the probability is 12/15 = **4/5**.

Also notice that no matter which vertex is chosen first, 4 of the 5 possible choices for the second vertex will be successes. This confirms our probability of **4/5**.

7. Since we are only interested in the order in which Annika (A), Billy (B) and Catherine (C) are called, we can ignore the other 8 students in Ms. McGinn's class. The different orders in which the triplets can be called are ABC, ACB, BCA, BAC, CAB and CBA. In two of these cases, BCA and BAC, Billy is called first. Therefore, the probability that Billy is the first triplet called to Ms. McGinn's desk is 2/6 = **1/3**.

Another way to look at it is the likelihood of being the first triplet called is the same for each triplet. Therefore, the probability of Billy being called first is **1/3**.

8. We are told that the probabilities of Alice, Bob and Carol each winning a game of chess is $P(A) = P(B) = P(C) = 1/3$. We are asked to determine the probability that Carol will win the championship in exactly 6 games. The only way Carol can win the championship after exactly 6 games have been played is if the third game she wins is the sixth game played. If her third win were to occur before the sixth game, no more games would be played following that win. We also know that the other three games can't be won by the same person, otherwise they would win the championship. So there must be two wins for Alice and one win for Bob, or one win for Alice and two wins for Bob. So the possible wins for the first five games are *ABBCC* or *AABCC*. Using the methodology from the video we see there are $5!/(2! \cdot 2!) = (5 \times 4 \times 3)/2 = 60/2 = 30$ orders in which the wins *ABBCC* (in any order) can occur and 30 orders in which the wins *AABCC* (in any order) can occur. That's a total of 60 sequences for the possible wins of the first five games. Since the probability of any of the three players winning each of those five games is 1/3, the probability that Carol will win the championship in exactly 6 games is $60 \times (1/3)^5 = 60/729 =$ **20/243**.