MATHCOUNTS® Minis November 2015 Activity Solutions

Warm-Up!

1a. One million is 1,000,000. Granted, we're not supposed to be writing anything, but just looking at this, I can see that $1{,}000{,}000 = 1000^2$, so the square of **999** (or 999^2) would be the largest square less than one million.

1b. This means we need the smallest positive three-digit integer that is 1 more than a multiple of 7. Let's build our smallest positive three-digit multiple of 7… it must be of the form 10_. Dividing 10 by 7 leaves a remainder of 3, and then making the units digit a 5 would make the situation such that 7 now evenly divides into 35. So 105 is the smallest positive three-digit multiple of 7 and **106** is the answer to the original question. (You may have gone a different route… many of us know 77 is a multiple of 7; and if we continue to add 7 or multiples of 7 we can find the number we're looking for. Adding 21 to 77 gives us 98, so 99 would give us a remainder of 1, but isn't a three-digit number; and then adding 7 more we get 106, so this is the smallest positive three-digit integer that is one more than a multiple of 7.)

1c. This is similar to the previous question. I know 999 is a multiple of 9, and that's pretty close to a four-digit number. In fact, if we add 5, we get **1004**, and we know that 1004 will leave a remainder of 5 when it's divided by 9, so this is our answer.

2. Since the three prime numbers are consecutive, their product should be close to the cube of the middle prime. The cube root of 2431 is about 13.4. Considering the consecutive primes 11, 13 and 17, we see that $11 \times 13 \times 17 = 2431$. The sum of the three primes is $11 + 13 + 17 =$ **41**.

3. We are looking for the units digit when we multiply all the odd numbers from 1 to 2015, inclusive. If we multiply $1 \times 3 \times 5 \times 7 \times 9$, and look only at the units digit, we see this product has a units digit of 5. Because the same digits are multiplied, the product of the odd numbers from 11 to 19 will have a units digit of 5, as will the product of the odd numbers from 21 to 29 and from 31 to 39, and so on. Therefore we know $1 \times 3 \times 5 \times \cdots \times 2009$ will have a units digit of 5, since 5 multiplied by itself, any number of times, will result in a units digit of 5. The only numbers left to multiply, then, are 2011, 2013 and 2015. Multiplying these three numbers will give us the same units digit as multiplying $1 \times 3 \times 5$, which is 5. The units digit for the product $1 \times 3 \times 5 \times \cdots \times 2015$, therefore, also will be **5**.

4. Since we want to maximize the difference, we should try the difference of a number in the nine-hundreds and a number in the one-hundreds.The greatest difference is $921 - 129 = 931 - 139 = 941 - 149 = 951 - 159 = 961 - 169 = 971 - 179 = 981 - 189 =$ **792**.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. The units digits of powers of 2 form the repeating pattern 2, 4, 8, 6. Since 2015 is three more than a multiple of 4, the units digit of 2^{2015} is 8, the third number in the pattern. Similarly, the units digits of powers of 7 form the repeating pattern 7, 9, 3, 1. This also is a pattern of four digits, so the units digit of 7^{2015} is 3, the third number in the pattern. Since $8 \times 3 = 24$, the units digit of $2^{2015} \times 7^{2015}$ is **4**.

6. Consider the four-digit number ABCD, where each letter represents a digit. We have A + B = C, B + C = D and C + D = 10A + B. If we rewrite the last equation as D = 10A + B – C, then we can substitute this for D in the second equation. We get B + C = 10A + B – C → 2C =10A → C = 5A. Since 1 is the only nonzero one-digit number that yields another one-digit number when multiplied by 5, it follows that A = 1 and C = 5. So 1 + B = 5 and B = 4. Also, we have 4 + 5 = D, so D = 9. Thus, the four-digit number is **1459**.

7. Similar to the video, let's look at the units digit of the cubes of the integers from 1 through 10. Respectively, these units digits will be 1, 8, 7, 4, 0, 6, 3, 2, 9 and 0. Since we are summing the units digit, we can eliminate the zeros and any pairs of integers that add to ten. Doing this we see that the units digit for the sum of the cubes of the integers from 1 through 10 will be 0. Since every batch of ten integers, 11 to 20, 21 to 30, and so on, results in a units digit of 0 when the cubes of the ten integers are summed, we know the sum of the cubes of the integers from 1 through 2010 will have a units digit of 0. Then the cubes of the integers from 2011 through 2015, the units digit will be the same as the sum of the cubes of the integers from 1 through 5. We know these units digits will be 1, 8, 7, 4 and 0. The sum is 20, so the units digit we are looking for will be **0**.

8. If 2015 + *a* = *b*, then *a* = *b* – 2015. Since *a* and *b* are both positive integers, we start subtracting 2015 from each of the possible *b* palindromes, beginning with 2112, which is the smallest palindrome greater than 2015. As the table shows, the first difference we obtain that also is a palindrome is 2772 – 2015 = **757**.

b	2112	2222	2332	2442	2552	2662	**2772**
a	97	207	317	427	537	647	**757**