MATHCOUNTS® Minis

May 2018 Activity Solutions

Warm-Up!

1. Since $4^x + 4^x + 4^x + 4^x = 4(4^x) = 2^2(2^2)^x = 2^2(2^{2x}) = 2^{2+2x}$, we can rewrite $4^x + 4^x + 4^x + 4^x = 2^8$ as follows: $2^{2+2x} = 2^8$. Now since the base for each power is 2, we can set the exponents equal to each other and solve for x. We get $2 + 2x = 8$, so $2x = 6$ and $x =$ **3**.

2. If $|x| = |x + 1|$, it follows that $x = x + 1$ or $x = -(x + 1)$. Solving the first equation leads to the contradiction $0 = 1$. Solving the second equation yields $x = -x - 1$, so $2x = -1$ and $x =$ **−1/2**. Substituting into the original equation, we see that $|-1/2| = |-1/2 + 1| \rightarrow 1/2 = |1/2| \rightarrow 1/2 = 1/2$.

3. Since $b(b^4 \times b^3)^2 = b(b^{4+3})^2 = b(b^7)^2 = b(b^{14}) = b^{15} = b^{3(5)}$, we can rewrite $b(b^4 \times b^3)^2 = b^{3x}$ as follows: $b^{3(5)} = b^{3x}$. It follows, then, that $x =$ **5**.

4. If $|x/2 - 6| = 20$, it follows that $x/2 - 6 = 20$ or $-(x/2 - 6) = 20$. Solving the first equation, we get $x/2 = 26$, so $x = \underline{52}$. Solving the second equation yields $6 - x/2 = 20$, so $-x/2 = 14$, and $x = \underline{-28}$. Since $|52/2 - 6| = |26 - 6| = |20| = 20$ and $|-28/2 - 6| = |-14 - 6| = |-20| = 20$, both solutions are valid. The positive difference between these solutions is $52 - (-28) = 52 + 28 =$ **80**.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Let's start by determining which perfect squares can obtained by multiplying two different numbers from 1 to 16, inclusive. Since $15 \times 16 = 240$, the perfect squares in question will all be less than 240. The perfect squares less than 240 are 1, 4, 9, 16, 25, 36 ,49, 64, 81, 100, 121, 144, 169, 196 and 225. The perfect squares that can not be obtained by multiplying two different numbers from 1 to 16, inclusive are 1, 25, 49, 81, 100, 121, 169, 196 and 225. We can, however, obtain the following products, which are perfect squares:

$4 = 1 \times 4$	$9 = 1 \times 9$	$36 = 3 \times 12 = 4 \times 9$
$16 = 1 \times 16 = 2 \times 8$	$64 = 4 \times 16$	$144 = 9 \times 16$

To avoid drawing two numbers whose product is a perfect square, Jillian must not draw the numbers 1, 2, 3, 4, 8, 9, 12 and 16. So the first eight numbers Jillian can draw without a pair whose product is a perfect square are 5, 6, 7, 10, 11, 13, 14, 15. There are 2 ways for her to draw the ninth and tenth numbers without a pair whose product is a perfect square:
Case 1: If she draws 2 and 3, that leaves the numbers 1, 4, 8, 9, 12 and 16.
Case 2: If she draws 8 and 12, that leaves the numbers 1, 2, 3, 4, 9 and 16.
In either case, if Jillian next draws the 1, 4, 9 or 16, she will have drawn a total of 11 numbers without a pair whose product is a perfect square. Any number drawn after that will result in a pair of numbers whose product is a perfect square, so the maximum number of slips that Jillian can draw is **11** slips.

6. Ordering the known values gives us 5, 15, 25, 35. Depending on the value of x, the median of the five integers could be 15, x or 25. The mean of the five numbers is $(5 + 15 + 25 + 35 + x) \div 5 = (80 + x) \div 5$. Since we are told that the median and mean are equal, we can write the following equations: $(80 + x) \div 5 = 15$, $(80 + x) \div 5 = x$ and $(80 + x) \div 5 = 25$. Solving the first equation, we get $80 + x = 75$, so $x = \underline{-5}$. Solving the second equation, we get $80 + x = 5x$, so $80 = 4x$, and $x = \underline{20}$. Solving the third equation, we get $80 + x = 125$, so $x = \underline{45}$. The sum of the three possible values of x is $-5 + 20 + 45 =$ **60**.

7. Consider the three functions $f(x) = |x - 2|$, $g(x) = |x - 4|$ and $h(x) = |x - 5|$. We know that

$f(x) = \begin{cases} 2 - x, \text{ for } x < 2 \\ x - 2, \text{ for } x \geq 2 \end{cases}$ $h(x) = \begin{cases} 4 - x, \text{ for } x < 4 \\ x - 4, \text{ for } x \geq 4 \end{cases}$ $g(x) = \begin{cases} 5 - x, \text{ for } x < 5 \\ x - 5, \text{ for } x \geq 5 \end{cases}$

It follows, then, that $|x - 2| + |x - 4| + |x - 5| = \begin{cases} 2 - x + 4 - x + 5 - x = 11 - 3x, \text{ for } x < 2 \\ x - 2 + 4 - x + 5 - x = 7 - x, \text{ for } 2 \leq x < 4 \\ x - 2 + x - 4 + 5 - x = x - 1, \text{ for } 4 \leq x < 5 \\ x - 2 + x - 4 + x - 5 = 3x - 11, \text{ for } x \geq 5 \end{cases}$

The minimum $y =$ **3** occurs when $x = 4$.

8. Recall that $n^m = 1$ in the following cases:

Case 1: for any m when $n = 1$
Case 2: for any n when $m = 0$
Case 3: for any even number m when $n = -1$

So, we need to solve $((2/3)x^2 - x - 2/3)^{(x^2 - 9x + 20)} = 1$ for three cases.

Case 1: $(2/3)x^2 - x - 2/3 = 1$
Solving for x, we get $(2/3)x^2 - x - 2/3 = 1 \rightarrow (2/3)x^2 - x - 5/3 = 0 \rightarrow 2x^2 - 3x - 5 = 0 \rightarrow (2x - 5)(x + 1) = 0$. So, $2x - 5 = 0$, and $x = 5/2$; or $x + 1 = 0$, and $x = -1$. Substituting 5/2 and -1 back into the original equation confirms that $\underline{5/2}$ and $\underline{-1}$ are both solutions.

Case 2: $x^2 - 9x + 20 = 0$
Solving for x, we get $x^2 - 9x + 20 = 0 \rightarrow (x - 4)(x - 5) = 0$. So, $x - 4 = 0$, and $x = 4$; or $x - 5 = 0$, and $x = 5$. Substituting 4 and 5 back into the original equation confirms $\underline{4}$ and $\underline{5}$ are both solutions.

Case 3: $(2/3)x^2 - x - 2/3 = -1$
Solving for x, we get $(2/3)x^2 - x - 2/3 = -1 \rightarrow (2/3)x^2 - x + 1/3 = 0 \rightarrow 2x^2 - 3x + 1 = 0 \rightarrow (2x - 1)(x - 1) = 0$. So, $2x - 1 = 0$, and $x = 1/2$; or $x - 1 = 0$, and $x = 1$. Substituting 1/2 and 1 back into the original equation confirms that 1/2 is not a solution, but $\underline{1}$ is a solution.

Thus, the sum of the solutions is $5/2 - 1 + 4 + 5 + 1 = (5 - 2 + 8 + 10 + 2)/2 =$ **23/2**.

9. If the number Larry tells Mary (or Jerry) is 1 or 10, then Jerry (or Mary) would immediately know the other's number is 2 or 9, respectively, and the given conversation would not occur. If the number Larry tells Mary is $\underline{2}$ or $\underline{9}$, and Jerry says he does not know Mary's number, then, logically, Mary can eliminate 1 or 10, respectively, and conclude that Jerry's number is 3 or 10, respectively. Finally, if the number Larry tells Mary is $\underline{3}$ or $\underline{8}$, she would not know if Jerry's number is 2 or 4, or 7 or 9, respectively. If Jerry then says he doesn't know Mary's number, then, logically, Mary can eliminate 2 or 9, respectively, and conclude that Jerry's number is 4 or 7, respectively. Therefore, for the given conversation to have occured, the sum of Mary's possible numbers is $2 + 3 + 8 + 9 =$ **22**.