MATHCOUNTS® Minis

May 2016 Activity Solutions

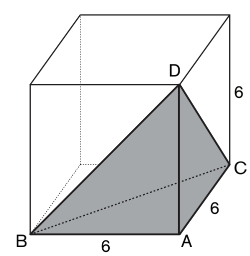
Warm-Up!

1. The area of the shaded region is the difference of the areas of the square and the circle. Since the square has sides of length 10 units, it follows that its area is $s^2 = 10^2 = 100$ units^2. The diameter of the circle is also 10 units, which means the radius is 5 units. The area of the circle then is $\pi r^2 = \pi(5^2) = 25\pi$ units^2. Therefore, the area of the shaded region is $100 - 25\pi$ unit^2 $\approx$ **21.46** units^2.

2. The border consists of four rectangles on each side of the square, with length 4 inches and width 1.5 inches, and four quarter circles at each corner with radius 1.5 inches. This make the border area equal to $4 \times (4 \times 1.5) + \pi(1.5)^2 = 24 + 2.25\pi \approx$ **31.1** in^2.s

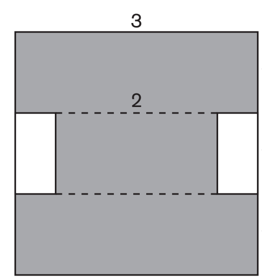
3. One face of the cube has an area of 81 units^2. This means the length of one side of the cube is $s = \sqrt{81} = 9$ units. The volume of the cube is $s^3 = 9^3 =$ **729** units^3.

4. Tetrahedron ABCD is shown here. The volume of the tetrahedron is $1/3 \times B \times h$, where B is the area of the base, and h is the height of the tetrahedron. The area of the base (ΔABC) is $½ \times 6 \times 6 = 18$ units^2. The height is 6 units. The volume of the tetrahedron is $1/3 \times 18 \times 6 = 36$ units^3.



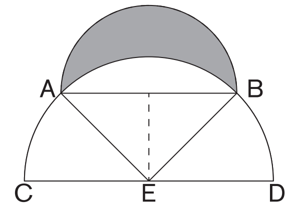
The Problems are solved in the **MATHCOUNTS® Mini** video.

Follow-up Problems

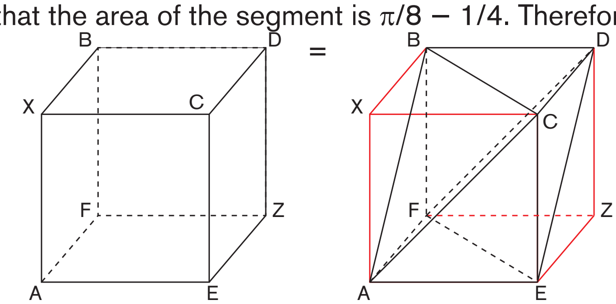


5. The upper and lower layer are each made of 9 unit cubes, so they have dimensions of $3 \times 3 \times 1$. The middle layer is made of 4 unit cubes, so it has dimensions $2 \times 2 \times 1$. The top face of the stack and the bottom face of the stack each have areas of $3 \times 3 = 9$. The four side faces, if we look at them straight on, appear as shown. The total area is the area of the the top and bottom rectangles, which are equal, plus the area of the middle rectangle, $2 \times 3 \times 1 + 2 \times 1 = 6 + 2 = 8$. The area of the top of the lower layer and the bottom of the upper layer are each $3 \times 3 - 2 \times 2 = 9 - 4 = 5$. So the total surface area for the cube stack is the top and bottom faces plus the four side faces plus the top and bottom of the lower and upper layers or $A = 2 \times 9 + 4 \times 8 + 2 \times 5 = 18 + 32 + 10 =$ **60** units^2.

6. Let AC be $2x$ and CD be x. The area of the semicircle with diameter AB is $2x^2\pi$. The area of the semicircle with diameter CB is $0.5x^2\pi$. The total area is $2.5x^2\pi$. Half the area (i.e., the two areas separated by segment CP) is $1.25x^2\pi$. Half the area of the semicircle with diameter AB is $x^2\pi$. Thus, half the area of the entire region is just 25% more than half of the area of the larger semicircle. An angle of 90° gives us an area of $x^2\pi$. So we need 25% of 90 or an additional 22.5°. The measure of angle ACP is therefore $90 + 22.5 =$ **112.5** degrees.



7. The area of the lune is the area of the smaller semicircle less the area of the segment of the larger semicircle intercepting arc AB and bounded by chord AB. Since the AB = 1, the area of the smaller semicircle is $\frac{1}{2} \times \pi \times r^2 = \frac{1}{2} \times \pi \times (\frac{1}{2})^2 = \pi/8$ units^2. The area of the segment of the larger semicircle bounded by chord AB is the area of sector of the larger semicircle intercepting arc AB less the area of ΔABE, as shown, where E is the midpoint of segment CD. Notice that BE = $\sqrt{2}/2$ because it is a radius of the larger semicircle whose diameter we are told is $\sqrt{2}$ units. The segment drawn from point E perpendicular to segment AB intersects the center of the smaller semicircle. Using the Pythagorean Theorem we see that the length of this segment is 1/2 units, which means point E also lies on the smaller circle. Since ∠AEB intercepts the diameter of the smaller semicircle, $m\angle AEB = 90°$. That means the area of the sector intercepting arc AB is 90/180 = 1/2 the area of the semicircle. The area of the sector is $\frac{1}{2} \times (\frac{1}{2} \times \pi \times r^2) = \frac{1}{2} \times (\frac{1}{2} \times \pi \times (\sqrt{2}/2)^2) = \frac{1}{2} \times (\frac{1}{4}\pi) = \pi/8$. The area of ΔAEB is $\frac{1}{2} \times b \times h = \frac{1}{2} \times 1 \times \frac{1}{2} = 1/4$ units^2. It follows that the area of the segment is $\pi/8 - 1/4$. Therefore, the area of the lune is $\pi/8 - (\pi/8 - 1/4) =$ $\pi/8 - \pi/8 + 1/4 =$ **1/4** units^2.



8. Starting with cube XBDCAFZE with side length 4 units, shown here, and removing tetrahedron XABC and tetrahedron ZDEF, to create the two equilateral triangular faces ABC and DEF, will leave us with our desired polyhedron. The volume of the cube itself is 4 × 4 × 4 = 64 units^3. The volume of the two congruent tetrahedrons are 1/3 × 1/2 × 4 × 4 × 4 = 10 2/3 units^3. So the volume of the polyhedron is 64 − 2 × 10 2/3 = 64 − 21 1/3 = **42 2/3** units^3.