MATHCOUNTS® Minis

March 2018 Activity Solutions

Warm-Up!

1. Since $400 = 256 + 128 + 16 = 2^8 + 2^7 + 2^4$, the sum of the exponents is $8 + 7 + 4 =$ **19**.

2. The base 10 equivalent of 11011000_2 is $2^7(1) + 2^6(1) + 2^5(0) + 2^4(1) + 2^3(1) + 2^2(0) + 2^1(0) + 2^0(0) = 64 + 16 + 8 = 216$. And the base 4 equivalent of 216 is $192 + 16 + 8 = 4^3(3) + 4^2(1) + 4^1(2) + 4^0(0) =$ **3120_4**.

3. The prime factorization of 240 is $2^4 \times 3 \times 5 = 2^3 \times 2 \times 3 \times 5$. If we wish to make the smallest perfect cube, we will need two more factors of 2, two more factors of 3, and two more factors of 5 to get $2^3 \times 2^3 \times 3^3 \times 5^3$. The value of k must be $2^2 \times 3^2 \times 5^2 = 4 \times 9 \times 25 =$ **900**.

4. In base six, the ones place is still the ones place, but the next place value is six, not ten. Thus, $53_6 = 6^1(5) + 6^0(3) = 33_{10}$. Since this amount base b equivalent requires three digits, we know that b must be smaller than six. Also, since the ones digit is 3, base b must be at least four. So b must be 4 or 5. The value of 113_4 is $4^2(1) + 4^1(1) + 4^0(3) = 16 + 4 + 3 = 23_{10}$, which does not equal 53_6. The value of 113_5 is $5^2(1) + 5^1(1) + 5^0(3) = 25 + 5 + 3 = 33_{10}$, which does equal 53_6. Thus, the value of b is **5**.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Since $2n$ is a perfect square, then n must have a factor of 2 to match the coefficient, and the remaining factors must multiply to a perfect square. Numbers that would work are 2, 8, 18, 32, 50, 72, and so on, or twice every square number. Similarly, if $3n$ is a perfect cube, then n must contain two factors of 3, and the remaining factors must multiply to a perfect cube. These numbers will all be nine times a perfect cube, or 9, 72, 243, and so on. The least number that appears in both lists is **72**. Checking, we see that if $n = 72$, then $2n = 144$, which is a perfecgt square, and $3n = 216$, which is a perfect cube.

6. In base 10, $AB_9 = 9^1(A) + 9^0(B) = 9A + B$. In base 10, $BA_7 = 7^1(B) + 7^0(A) = 7B + A$. We are told that $AB_9 = BA_7$, so we can write $9A + B = 7B + A \rightarrow 8A = 6B \rightarrow A/B = 6/8 = 3/4$. Now $A/B = 3/4$ and the only values of A and B that work are A = 3 and B = 4. (Note that if A = 6, then B = 8. But the digit 8 does not exist in base 7, since base 7 numbers are expressed using the digits 0 through 6). Thus, in base 10, $AB_9 = 9(3) + 4 = 27 + 4 =$ **31**, and $BA_7 = 7(4) + 3 = 28 + 3 =$ **31**.

7. If $2015 + a = b$, then $a = b - 2015$. Since a and b are both positive integers, we start subtracting 2015 from each of the possible b palindromes, beginning with 2112, which is the smallest palindrome greater than 2015. As the table shows, the first difference we obtain that also is a palindrome is $2772 - 2015 =$ **757**.

b	2112	2222	2332	2442	2552	2662	**2772**
a	97	207	317	427	537	647	**757**

8. Since 5 is a digit in one of the base b numbers, we know $b > 5$. Notice that 441 is a perfect square in base 10, so if we assume $b = 10$, then $n^2 = 441$, and $n = 21$. That means $(n - 2)^2 = 19^2 = 361$. But we are told that $(n - 2)^2 = 351_b$. So, $b \neq 10$, and since $361 > 351$, we conclude that $6 < b < 10$. Since 361 is just a little more than 351, let's try $b = 9$. We have $441_9 = 9^2(4) + 9^1(4) + 9^0(4) = 324 + 36 + 1 = 361_{10} = 19^2$, and $19 = n$. Then we have $351_9 = 9^2(3) + 9^1(5) + 9^0(1) = 243 + 45 + 1 = 289_{10} = 17^2$, and $17 = n - 2$. Therefore, $b =$ **9**.