Warm-Up!

1. Solve the equation $\sqrt{x + 4} = 3$ by squaring both sides. We get $x + 4 = 9$, so $x = 5$.

2. When we expand the given product we get $(a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$.

3. If we subtract the second equation from the first equation we get
   
   $$u + v + w + x + y + z = 45$$
   $$- (v + w + x + y + z = 21)$$
   $$u = 24$$

4. Let $x$ represent the first number and $y$ represent the second number. We are told that $x + y = 6$ and $xy = 7$. We are asked to find the sum of the reciprocals of the two numbers, $1/x + 1/y$, which can be rewritten as $(y + x)/xy$. Substituting, we have $(y + x)/xy = 6/7$.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Since we ultimately want the product of the three integers, let’s start by multiplying the products of the pairs we are given. We have $(xy)(yz)(xz) = 4 \times 18 \times 50 \rightarrow x^2y^2z^2 = 3600 \rightarrow (xyz)^2 = 3600.$ Now if we take the square root of each side, we get $xyz = 60$, since $x$, $y$ and $z$ are positive numbers.

6. We are told that $xyz = 45$ and $1/x + 1/y + 1/z = 1/5$. We can rewrite the left side of the second equation using a common denominator to get $(yz/xyz) + (xz/xyz) + (xy/xyz) = 1/5 \rightarrow (xy + xz + yz)/xyz = 1/5$. But we know that $xyz = 45$, so we have $(xy + xz + yz)/45 = 1/5 \rightarrow xy + xz + yz = 9$. If the sum of the three products $xy$, $xz$ and $yz$ is 9, then their mean is $9/3 = 3$.

7. Let’s start by cubing each side of the given equation to get
   
   $$\left( a + \frac{1}{a} \right)^3 = 3^3$$
   $$\left( a + \frac{1}{a} \right)^2 \left( a + \frac{1}{a} \right) = 27$$
   $$\left( a^2 + 2 + \frac{1}{a^2} \right) \left( a + \frac{1}{a} \right) = 27$$
   $$a^3 + a + 2a + \frac{2}{a} + \frac{1}{a} + \frac{1}{a^3} = 27$$
   $$a^3 + 3a + \frac{3}{a} + \frac{1}{a^3} = 27.$$
If we factor a 3 out of the two middle terms of the expression on the left-hand side of the equation, we have
\[ a^3 + 3\left(a + \frac{1}{a}\right) + \frac{1}{a^3} = 27. \]

Since we know that \(a + \frac{1}{a} = 3\), we can substitute and simplify to get
\[
\begin{align*}
   a^3 + 3(3) + \frac{1}{a^3} &= 27 \\
   a^3 + 9 + \frac{1}{a^3} &= 27 \\
   a^3 + \frac{1}{a^3} &= 18.
\end{align*}
\]

8. First, let's multiply the two given equations together
\[
\left(\frac{x}{y} + 1\right)
\left(\frac{y}{z} + 1\right) = 2\left(\frac{1}{2}\right)
\]
\[
xy + \frac{x}{z} + 1 + \frac{1}{yz} = 1,
\]
\[
xy + \frac{x}{z} + \frac{1}{yz} = 0.
\]

Multiplying through by \(z\), we get the equation
\[ xyz + x + \frac{1}{y} = 0. \]

Substituting in and solving for \(xyz\), we find that
\[ xyz + 2 = 0 \]
\[ xyz = -2. \]