January 2015 Activity Solutions

Warm-Up!

1. Let's make an organized list to keep track of the parallelograms as we find them. We will categorize each parallelogram based on the number of rows and columns it spans. There are 4 parallelograms that span one row and one column. There are 2 parallelograms that span one row and two columns. There are also 2 parallelograms that span two rows and one column. Finally, there is 1 parallelogram that spans two rows and two columns. That brings the total number of parallelograms in the figure to 4 + 2 + 2 + 1 = **9** parallelograms.

2. Consider five dresser drawers labeled from top to bottom A through E. First, if exactly one drawer is open, the 5 possibilities are drawer A, B, C, D or E. Next, if exactly two drawers are open, the 6 possibilities are drawers A and C, A and D, A and E, B and D, B and E or C and E. Finally, there is only 1 possible way to have exactly three drawers open, drawers A, C and E. That's a total of 5 + 6 + 1 = **12** ways in which one of more of the drawers can be opened to access the contents of each open drawer.

3. Let a, b and $c = 5$ be the side lengths of the triangle. Since the sides have lengths that are integers no greater than 5 units, the possible values of a and b are 1, 2, 3, 4 and 5. By definition, we know that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. In other words, $a + b > 5$. The a-b-c triples for the **9** possible distinct triangles are 1-5-5, 2-4-5, 2-5-5, 3-3-5, 3-4-5, 3-5-5, 4-4-5, 4-5-5 and 5-5-5.

The Problem is solved in the MATHCOUNTS® Mini video.

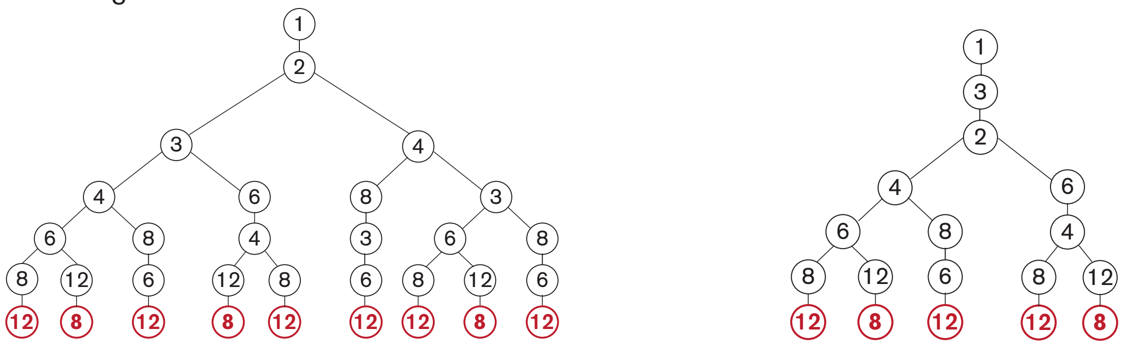
Follow-up Problems

4. It seems that organizing and listing possibilities has worked for each problem thus far, so let's try it again. First, notice that the sum $m^2 + n$ must be less than 31. That means the largest possible value for m is 5 since 6^2 is 36, which is greater than 31. Now let's rewrite the inequality in the form $n < 31 - m^2$ and examine the possible values of n when m is 1, 2, 3, 4 or 5.

- If $m = 1$, we have $n < 31 - 1^2 \rightarrow n < 30$. The integer n can be any integer from 1 to 29, inclusive.
- Next, if $m = 2$, we have $n < 31 - 2^2 \rightarrow n < 27$. In this case, n can be any integer from 1 to 26, inclusive.
- If $m = 3$, we have $n < 31 - 3^2 \rightarrow n < 22$. Here n can take the value of any integer from 1 to 21, inclusive.
- If $m = 4$, we have $n < 31 - 4^2 \rightarrow n < 15$. In this case, n can be any integer from 1 to 14, inclusive.
- Finally, if $m = 5$, we have $n < 31 - 5^2 \rightarrow n < 6$. The integer n can be any integer from 1 to 5, inclusive.

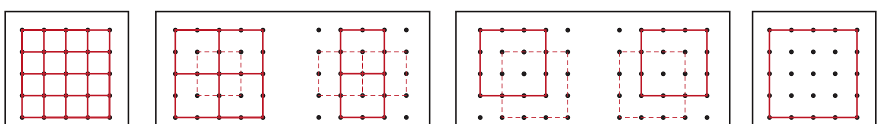
Therefore, the number of pairs of integers that satisfy the inequality is 29 + 26 + 21 + 14 + 5 = **95** pairs.

5. Using the technique employed in the video, we will use tree diagrams to aid in counting the arrangements. Any arrangement that meets the given criteria must begin with 1 since it is a divisor of every number in the set. After 1, the second number could be 2 or 3. The tree diagrams show the possible arrangements for these two cases.



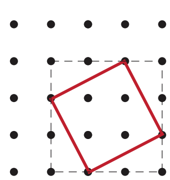
There are 9 ways to order the numbers beginning with 1, 2, ..., and there are 5 ways to order them beginning with 1, 3,... There are a total of 9 + 5 = **14** ways this set of numbers can be ordered.

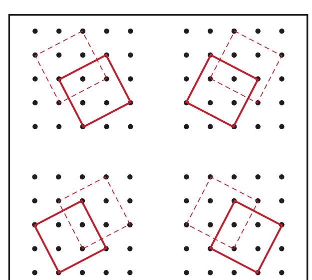
6. This problem is a bit trickier than the parallelogram problem from the video. First, we can't arbitrarily select two horizontal lines and two vertical lines to form a square. Once you select two horizontal lines the vertical lines have to be selected in such a way that the horizontal length is the same as the vertical length. Another consideration is that squares can be formed in the grid by combining a pair of horizontal lines with a pair of vertical lines and by combining a pair of diagonal lines with a positive slope with a pair of diagonal lines with a negative slope. Let's first determine how many squares can be created using only horizontal and vertical lines. We begin by making an organized list. There are 16 1-by-1 squares. There are nine 2-by-2 squares. There are four 3-by-3 squares. And there is one 4-by-4 square. These squares are shown below.



1 × 1 16 squares | 2 × 2 9 squares | 3 × 3 4 squares | 4 × 4 1 square

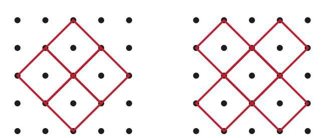
Now we will organize and list the possible diagonal squares. The figure below on the left shows a diagonal square (solid line) inside a 3-by-3 region (dotted line) of the grid. To determine the side lengths of the square we can use the Pythagorean Theorem. Notice the right triangles in each corner of the 3-by-3 region. Each of these triangles has a short leg with length 1 unit, a long leg with length 2 units and a hypotenuse that is a side of the diagonal square. We know the length of the hypotenuse is $c = \sqrt{(a^2 + b^2)} \rightarrow c = \sqrt{(1^2 + 2^2)} \rightarrow c = \sqrt{5}$. We can form 7 more squares of this size on our grid, as shown below on the right.



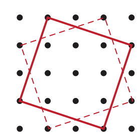


$\sqrt{5} \times \sqrt{5}$
8 squares

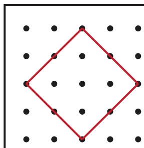
But we need to determine all the possible side lengths for these diagonal squares to ensure that we don't miss any. To find all the possible side lengths, consider that the side of the square will be the hypotenuse of a right triangle and we can write $c = \sqrt{(a^2 + b^2)}$, where *c* is the length of the hypotenuse and *a* and *b* are the lengths of the shorter and longer legs of the triangle, respectively. If $a = 1$, we can form right triangles such that $b = 1$, $b = 2$ or $b = 3$. It doesn't work if $b = 4$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $c = \sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $c = \sqrt{(1^2 + 3^2)} = \sqrt{10}$. If $a = 2$, we can form right triangles such that $b = 1$ or $b = 2$. Again, it doesn't work if $b = 3$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(2^2 + 1^2)} = \sqrt{5}$ and $c = \sqrt{(2^2 + 2^2)} = 2\sqrt{2}$. Below are the squares that can be formed with sides of lengths $\sqrt{2}$ units, $\sqrt{10}$ units and $2\sqrt{2}$ units (the 8 squares with sides of length $\sqrt{5}$ units were shown previously).



$\sqrt{2} \times \sqrt{2}$
9 squares



$\sqrt{10} \times \sqrt{10}$
2 squares



$2\sqrt{2} \times 2\sqrt{2}$
1 square

Finally, that brings the total number of squares to $16 + 9 + 4 + 1 + 8 + 9 + 2 + 1 =$ **50** squares.