MATHCOUNTS® Minis — January 2014 Activity Solutions

Warm-Up!

1. Using the distance formula, we have $d = \sqrt{(3+1)^2 + (7-10)^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = \mathbf{5}$.

2. The slope of a line perpendicular to line *k* will have a slope that is the opposite reciprocal of that of line *k*. Using the slope formula for line *k*, we get (−5 −8)/(2 + 1) = −13/3. Therefore, the slope of a line perpendicular to line *k* is **3/13**.

3. The perpendicular bisector of the segment with endpoints (2, 3) and (6, 9) intersects this segment at its midpoint. The coordinates of the midpoint of the segment are ((2 + 6)/2, (3 + 9)/2) = (4, 6). We also know that the slope of the perpendicular bisector of the segment is the opposite reciprocal of that of the segment. Using the slope formula for the segment, we get (9 − 3)/(6 − 2) = 6/4 = 3/2. Therefore, the slope of the perpendicular bisector is −2/3. With the slope of the perpendicular bisector and the coordinates of a point on this line, we can derive an equation for the line. Substituting for *m*, *x* and *y* in the point-slope form $y - y_1 = m(x - x_1)$, we have $y - 6 = (-2/3)(x - 4) \rightarrow y - 6 = (-2/3)x + 8/3 \rightarrow (2/3)x + y = 26/3$. Because we need the equation to be in the form x + By = C, which has a leading coefficient of 1, we multiply this equation by 3/2, to get $x + (3/2)y = 13$. Thus, C = **13**.

4. The segment with endpoints (−2, 3) and (6, −7) has midpoint ((−2 + 6)/2, (3 − 7)/2) = (2, −2). This point is equidistant from the endpoints of the segment, as are all points on a line through this point and perpendicular to this segment. The slope of the segment is (−7 − 3)/(6 + 2) = −10/8 = −5/4. That means the slope of the line perpendicular to the segment through (2, −2) is 4/5. Substituting *m*, *x* and *y* into $y - y_1 = m(x - x_1)$, we have $y + 2 = (4/5)(x - 2) \rightarrow y + 2 = (4/5)x - 8/5 \rightarrow (-4/5)x + y = -18/5$. Again, we need a leading coefficient of 1, so we multiply by −5/4 to get $x - (5/4)y = 9/2$. Thus, B = **−5/4**.

The Problems are solved in the video.

Follow-up Problems

5. From the video we know that any point that is equidistant from (3, −1) and (7, −9) will be on the perpendicular bisector of the segment joining these two points. So if we find an equation for that line we can determine where it will intersect the line given by $x + y = 3$. The midpoint of the segment from (3, −1) to (7, −9) is ((3 + 7)/2, (−1 − 9)/2) = (5, −5). The slope of the segment is (−1 + 9)/(3 − 7) = 8/(−4) = −2. It follows, then, that the slope of the perpendicular bisector is 1/2. Substituting for *m*, *x* and *y* in the point slope form, we have $y + 5 = (1/2)(x - 5) \rightarrow y + 5 = (1/2)x - 5/2 \rightarrow -(1/2)x + y = -15/2$. Multiplying by 2 to eliminate the fractions, we have $-x + 2y = -15$. To determine the intersection of the two lines given by $x + y = 3$ and $-x + 2y = -15$, let's add the two equations as follows: $(x + y) + (-x + 2y) = 3 - 15 \rightarrow 3y = -12 \rightarrow y = -4$. Substituting this value into $x + y = 3$, we get $x - 4 = 3 \rightarrow x = 7$. Thus, the point on $x + y = 3$ that is equidistant from (3, −1) and (7, −9) has coordinates **(7, −4)**.

6. Let's employ a technique similar to the one used in the video. We know that any point that is equidistant from $(-2, 4)$ and $(6, 12)$ will be on the perpendicular bisector of the segment joining these two points. The midpoint of the segment joining $(-2, 4)$ and $(6, 12)$ is $((-2 + 6)/2, (4 + 12)/2) = (2, 8)$. The segment has slope $(12 - 4)/(6 + 2) = 8/8 = 1$. Therefore, the perpendicular bisector intersects the segment at $(2, 8)$ and has slope -1. We can substitute this information into the point-slope form to determine the equation of the perpendicular bisector: $y - 8 = -1(x - 2) \rightarrow y - 8 = -x + 2 \rightarrow y = 10 - x$. Now we know that the two points that are $5\sqrt{2}$ units from both $(-2, 4)$ and $(6, 12)$ are on the line given by $y = 10 - x$. We can use the distance formula to find the coordinates of a point $(x, 10 - x)$ that is $5\sqrt{2}$ units from $(6, 12)$. Substituting, we have $\sqrt{(x-6)^2 + (10 - x - 12)^2} = 5\sqrt{2} \rightarrow (x - 6)^2 + (-x - 2)^2 = 50 \rightarrow x^2 - 12x + 36 + x^2 + 4x + 4 = 50 \rightarrow 2x^2 - 8x + 40 = 50 \rightarrow 2x^2 - 8x - 10 = 0 \rightarrow x^2 - 4x - 5 = 0 \rightarrow (x + 1)(x - 5) = 0$. So $x + 1 = 0 \rightarrow x = -1$ and $x - 5 = 0 \rightarrow x = 5$. Substituting, we see that when $x = -1$, $y = 10 + 1 = 11$ and when $x = 5$, $y = 10 - 5 = 5$. So, the two points that are $5\sqrt{2}$ units from $(6, 12)$, and subsequently $5\sqrt{2}$ units from $(-2, 4)$, are **(−1, 11)** and **(5, 5)**.

7. The circumcenter of the triangle with vertices A(0, 0), B(6, 0) and C(4, 2) is equidistant from each vertex since the segments from the center to each of these vertices are radii of the circle. It also must be the point at which the perpendicular bisectors of the sides of the triangle intersect. We need only the equations of two of the three bisectors to determine this point of intersection. Side AC has midpoint $((6 + 0)/2, (0 + 0)/2) = (3, 0)$. Because side AC is a horizontal line, it follows that its perpendicular bisector is the vertical line through $(3, 0)$ given by $x = 3$. Side BC has midpoint $((4 + 6)/2, (2 + 0)/2) = (5, 1)$ and slope $(2 - 0)/(4 - 6) = 2/(-2) = -1$. It follows, then that the perpendicular bisector of side BC has slope 1 and is given by $y - 1 = 1(x - 5) \rightarrow y = x - 4$. Substituting, we see that when $x = 3$, $y = 3 - 4 = -1$. Therefore, the circumcenter of $\triangle$ABC is **(3, −1)**.

8. We solved the previous problems using the properties of a segment bisector, but for this problem, we will use the properties of an *angle bisector*. Recall that an angle bisector is the ray originating at the vertex of an angle that divides the angle into two congruent angles. By theorem, every point on the line that bisects a given angle is equidistant from the sides of the angle.
We are told that the point $(x, 2x)$, which is on the line $y = 2x$, is equidistant from both sides of the angle formed by segments AB and BC. It follows, then that the $(x, 2x)$ must be a point on the line that bisect $\angle$ABC. The slope of side AB is $(3 - 0)/(5 - 2) = 3/3 = 1$, and the slope of side BC is $(4 - 0)/(-2 - 2) = 4/(-4) = -1$. So, side AB is perpendicular to side BC and $m\angle ABC = 90°$. That means that the bisector of $\angle$ABC is the vertical line through $(2, 0)$ given by $x = 2$. Thus, $x =$ **2**.