MATHCOUNTS® Minis February 2015 Activity Solutions

Warm-Up!

1. If we subtract the second equation from the first equation we get

$$\begin{array}{r} u + v + w + x + y + z \;\; = 45 \\ \underline{-\;(\;v + w + x + y + z \;\; = 21\;)} \\ u = \mathbf{24} \end{array}$$

2. When we expand the given product, we get $(x + 1)(y + 1) = \boldsymbol{xy + x + y + 1}$.

3. When we expand the given product, we get $(a + b)(a + b) = a^2 + ab + ab + b^2 = \boldsymbol{a^2 + 2ab + b^2}$.

4. For the first number, let x and y be the numerator and denominator, respectively. For the second number, let u and v be the numerator and denominator, respectively. We are told that $\frac{x}{y} + \frac{u}{v} = 6$, which can be rewritten as $\frac{xv + uy}{yv} = 6 \rightarrow xv + uy = 6yv$. We are also told that $\frac{x}{y} \times \frac{u}{v} = 7$, which can be rewritten as $\frac{xu}{yv} = 7 \rightarrow xu = 7yv$. We are asked to find the sum of reciprocals of the two numbers, $\frac{y}{x} + \frac{v}{u}$, which can be rewritten as $\frac{uy + xv}{xu}$. Substituting, we have $\frac{uy + xv}{xu} = \frac{6yv}{7yv} = \mathbf{\frac{6}{7}}$.

The Problem is solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Since we ultimately want the product of the three integers, let's start by multiplying the products of the pairs we are given. We have $(xy)(yz)(xz) = 4 \times 18 \times 50 \rightarrow x^2y^2z^2 = 3600 \rightarrow (xyz)^2 = 3600$. Now if we take the square root of each side, we get $xyz = \mathbf{60}$, since x, y and z are positive numbers.

6. We are told that $xyz = 45$ and $1/x + 1/y + 1/z = 1/5$. We can rewrite the left side of the second equation using a common denominator to get $(yz/xyz) + (xz/xyz) + (xy/xyz) = 1/5 \rightarrow$ $(xy + xz + yz)/xyz = 1/5$. But we know that $xyz = 45$, so we have $(xy + xz + yz)/45 = 1/5 \rightarrow$ $xy + xz + yz = 9$. If the sum of the three products xy, xz and yz is 9, then their mean is $9/3 = \mathbf{3}$.

7. Let's start by cubing each side of the given equation to get

$$\left(a + \frac{1}{a}\right)^3 = 3^3$$

$$\left(a + \frac{1}{a}\right)^2\left(a + \frac{1}{a}\right) = 27$$

$$\left(a^2 + 2 + \frac{1}{a^2}\right)\left(a + \frac{1}{a}\right) = 27$$

$$a^3 + a + 2a + \frac{2}{a} + \frac{1}{a} + \frac{1}{a^3} = 27$$

$$a^3 + 3a + \frac{3}{a} + \frac{1}{a^3} = 27.$$

If we factor a 3 out of the two middle terms of the expression on the left-hand side of the equation, we have

$$a^3 + 3\left(a + \frac{1}{a}\right) + \frac{1}{a^3} = 27.$$

Since we know that $a + \frac{1}{a} = 3$, we can substitute and simplify to get

$$a^3 + 3(3) + \frac{1}{a^3} = 27$$

$$a^3 + 9 + \frac{1}{a^3} = 27$$

$$a^3 + \frac{1}{a^3} = \mathbf{18}.$$

8. We are told that $a + (1/a) = 6$ and asked to determine the value of $a^4 + (1/a^4)$. Squaring the first equation yields $(a + (1/a))^2 = 6^2 \rightarrow a^2 + 2 + (1/a^2) = 36 \rightarrow a^2 + (1/a^2) = 34$. Now squaring each side of this equation, we get $(a^2 + (1/a^2))^2 = 34^2 \rightarrow a^4 + 2 + (1/a^4) = 1156$. Therefore, $a^4 + (1/a^4) =$ **1154**.

9. Cross-multiplying, we get $c(b + 4) = 9(b + 7) \rightarrow bc + 4c = 9b + 63 \rightarrow bc + 4c - 9b = 63$. We are looking for all possible pairs of integers b and c such that $bc + 4c - 9b$ equals 63. It would be helpful if the equation were written as a product of two expressions equal to some integer. We would have something similar to the following: (b expression) × (c expression) = integer. To achieve this, we start by rewriting the equation as $c(b + 4) - 9b = 63$. Notice that if, instead of $-9b$, we had $-9(b + 4)$, we would be able to factor the left-hand side of the equation to get $(b + 4)(c - 9)$. In fact, subtracting 36 from both sides of the equation $c(b + 4) - 9b = 63$ yields $c(b + 4) - 9b - 36 = 63 - 36 \rightarrow c(b + 4) - 9(b + 4) = 27 \rightarrow (b + 4)(c - 9) = 27$. Now, we need only use each of the factor pairs of 27 to determine all pairs of integers b and c that satisfy the equation.

***b* + 4**	***c* – 9**	***b***	***c***
1	27	–3	36
–1	–27	–5	–18
27	1	23	10
–27	–1	–31	8
3	9	–1	18
–3	–9	–7	0
9	3	5	12
–9	–3	–13	6

As the table shows, there are **8** pairs of integers (b, c) that satisfy the original equation.

NOTE: The same solution results solving algebraically as follows:
$(b + 7)/(b + 4) = c/9 \rightarrow [(b + 4) + 3]/(b + 4) = c/9 \rightarrow (b + 4)/(b + 4) + 3/(b + 4) = c/9 \rightarrow$
$1 + 3/(b + 4) = c/9 \rightarrow 3/(b + 4) = c/9 - 1 \rightarrow 3/(b + 4) = c/9 - 9/9 \rightarrow 3/(b + 4) = (c - 9)/9$.
Cross-multiplying, then, yields $(b + 4)(c - 9) = 3(9) \rightarrow (b + 4)(c - 9) = 27$.