MATHCOUNTS® Minis February 2017 Activity Solutions

Warm-Up!

1. There are three different sized squares in the figure, 1 × 1, 2 × 2 and 3 × 3. There are <u>9</u> squares of size 1 × 1, <u>4</u> of size 2 × 2 and <u>1</u> of size 3 × 3, for a total of 9 + 4 + 1 = **14** squares.

2. Let's start by maximizing the number of dimes and examine the various cases as we reduce the number of dimes.

Case 1: Using two dimes, Jamie can make 26¢ in <u>2</u> ways:
10¢ + 10¢ + 5¢ + 1¢ = 26¢
10¢ + 10¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢

Case 2: Using one dime, Jamie can make 26¢ in <u>2</u> ways:
10¢ + 5¢ + 5¢ + 5¢ + 1¢ = 26¢
10¢ + 5¢ + 5¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢

Case 3: Using no dimes, Jamie can make 26¢ in <u>1</u> way:
5¢ + 5¢ + 5¢ + 5¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢

Based on these three cases, we see that given 2 dimes, 4 nickels and 8 pennies, Jamie can make 26¢ in 2 + 2 + 1 = **5** ways

3. Consider five dresser drawers labeled from top to bottom A through E. Let's start by considering the possibility when opening one drawer and examine the various cases as we increase the number of drawers opened.

Case 1: If exactly one drawer is open, the 5 possibilities are drawer A, B, C, D or E.

Case 2: If exactly two drawers are open, the 6 possibilities are drawers A and C, A and D, A and E, B and D, B and E or C and E.

Case 3: There is only 1 possible way to have exactly three drawers open, drawers A, C and E.

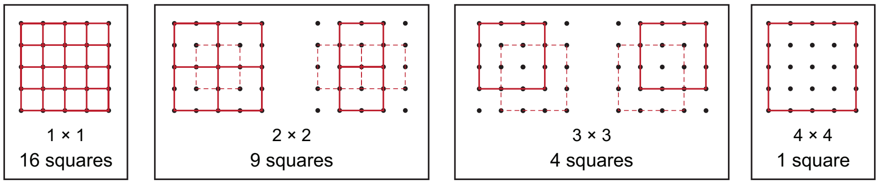
It is not possible to open more than three drawers. Based on these three cases, we see there is a total of 5 + 6 + 1 = **12** ways in which one of more of the drawers can be opened to access the contents of each open drawer.

4. Let a, b and $c = 5$ be the side lengths of the triangle. Since the sides have lengths that are integers no greater than 5 units, the possible values of a and b are 1, 2, 3, 4 and 5. By definition, we know that the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. In other words, $a + b > 5$. The a-b-c triples for the **9** possible distinct triangles are 1-5-5, 2-4-5, 2-5-5, 3-3-5, 3-4-5, 3-5-5, 4-4-5, 4-5-5 and 5-5-5.

The Problems are solved in the MATHCOUNTS® Mini video.

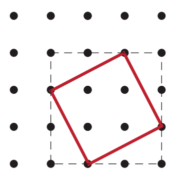
Follow-up Problems

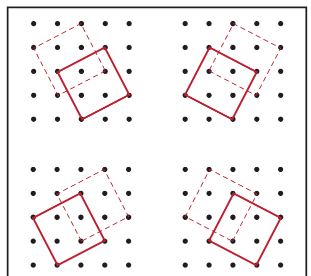
5. This problem is a bit trickier than the rectangle problem from the video. Squares can be formed in the grid by combining a pair of horizontal lines with a pair of vertical lines and by combining a pair of diagonal lines with a positive slope with a pair of diagonal lines with a negative slope. Let's first determine how many squares can be created using only horizontal and vertical lines. We begin by making an organized list: 16 of the squares are 1 × 1, 9 are 2 × 2, 4 are 3 × 3, 1 is 4 × 4. These squares are shown below.



1 × 1 16 squares | 2 × 2 9 squares | 3 × 3 4 squares | 4 × 4 1 square

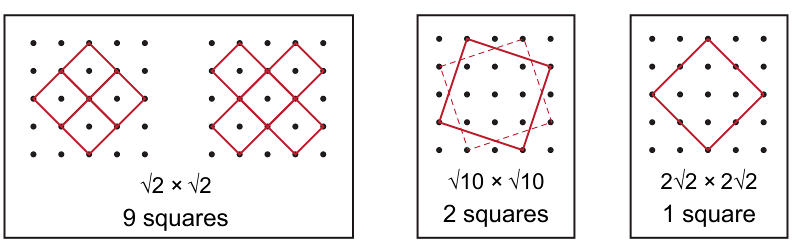
Now we will organize and list the possible diagonal squares. The figure below on the left shows a diagonal square (solid line) inside a 3 × 3 region (dotted line) of the grid. To determine the side lengths of the square we can use the Pythagorean Theorem. Notice the right triangles in each corner of the 3 × 3 region. Each of these triangles has a short leg with length 1 unit, a long leg with length 2 units and a hypotenuse that is a side of the diagonal square. We know the length of the hypotenuse is $c = \sqrt{(a^2 + b^2)} \rightarrow c = \sqrt{(1^2 + 2^2)} \rightarrow c = \sqrt{5}$. We can form 8 squares of this size on our grid, as shown below on the right.





$\sqrt{5} \times \sqrt{5}$
8 squares

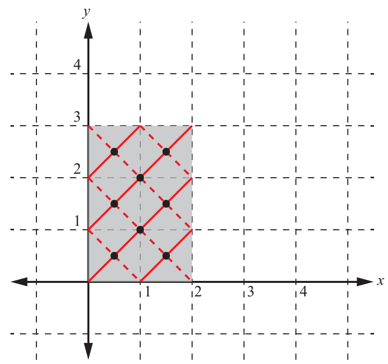
But we need to determine all the possible side lengths for these diagonal squares to ensure that we don't miss any. To find all the possible side lengths, consider that the side of the square will be the hypotenuse of a right triangle and we can write $c = \sqrt{(a^2 + b^2)}$, where c is the length of the hypotenuse and a and b are the lengths of the shorter and longer legs of the triangle, respectively. If $a = 1$, we can form right triangles such that $b = 1$, $b = 2$ or $b = 3$. It doesn't work if $b = 4$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $c = \sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $c = \sqrt{(1^2 + 3^2)} = \sqrt{10}$. If $a = 2$, we can form right triangles such that $b = 1$ or $b = 2$. Again, it doesn't work if $b = 3$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(2^2 + 1^2)} = \sqrt{5}$ and $c = \sqrt{(2^2 + 2^2)} = 2\sqrt{2}$. So, in addition to the 8 diagonal squares of side length $\sqrt{5}$ units, shown previously, there are 9 of side length $\sqrt{2}$, 2 with side length $\sqrt{10}$ and 1 of side length $2\sqrt{2}$, as shown.



$\sqrt{2} \times \sqrt{2}$ 9 squares | $\sqrt{10} \times \sqrt{10}$ 2 squares | $2\sqrt{2} \times 2\sqrt{2}$ 1 square

That brings the total number of squares to 16 + 9 + 4 + 1 + 8 + 9 + 2 + 1 = **50** squares.

6. The easiest way to count the intersections inside the shaded area is to draw the lines. The lines that pass through the shaded region with a slope of 1 and an integer y-intercept are $y = x - 1$, $y = x$, $y = x + 1$ and $y = x + 2$. The lines that pass through the shaded region with a slope of -1 and integer y-intercept are $y = -x + 1$, $y = -x + 2$, $y = -x + 3$ and $y = -x + 4$. We can see from the figure below, there are **8** intersections in the shaded region.



7. Let's start with the case in which the sum includes five 1. Then we'll reduce the number of 1s while listing the sums for each case.

Case 1: In the case of five 1s, there is only 1 sum: 13 + 1 + 1 + 1 + 1 + 1.

Case 2: In the case of four 1s, the other two odd numbers must add up to 14. There are three possibilities: 11 + 3, 9 + 5 and 7 + 7. This case yields 3 sums.

Case 3: In the case of three 1s, the other three odd numbers must add up to 15. There are three possibilities: 9 + 3 + 3, 7 + 5 + 3 and 5 + 5 + 5. This case yields 3 sums.

Case 4: In the case of two 1s, the other four odd numbers must add up to 16. There are two possibilities: 7 + 3 + 3 + 3 and 5 + 5 + 3 + 3. This case yields 2 sums.

Case 5: In the case of one 1, the other five odd numbers must add up to 17. There is one possibility: 5 + 3 + 3 + 3 + 3. This case yields 1 sum.

Case 6: In the case of no 1s, there is one possibility with six odd numbers that add up to 18: 3 + 3 + 3 + 3 + 3 + 3. This case yields 1 sum.

These six cases yield a total of 1 + 3 + 3 + 2 + 1 + 1 = **11** sums.

8. Let's start with the case of two blue marbles, since this is the fewest there could be, and add one blue marble for each new case.

Case 1: The two blue and three green marbles can be arranged the following ways: BBGGG, GBBGG, GGBBG, GGGBB. This case yeilds 4 arrangements.

Case 2: With three blue and two green marbles, all three blue must be together. We can arrange the marbles in the following ways: BBBGG, GBBBG, GGBBB. This case yeilds 3 arrangements.

Case 3: With four blue marbles and one green, the blue marbles can either be in a group of four or two groups of two. The marbles can be arranged in the following ways: BBBBG, BBGBB or GBBBB. This case yields 3 arrangements.

Case 4: Finally, all the marbles could be blue: BBBBB. This case yields 1 arrangement.

The total of all the cases is 4 + 3 + 3 + 1 = **11** arrangements.