MATHCOUNTS® Minis December 2014 Activity Solutions

Warm-Up!

1. Since $400 = 256 + 128 + 16 = 2^8 + 2^7 + 2^4$, the sum of the exponents is $8 + 7 + 4 =$ **19**.

2. The base 10 equivalent of 11011000_2 is $2^7(1) + 2^6(1) + 2^5(0) + 2^4(1) + 2^3(1) + 2^2(0) + 2^1(0) + 2^0(0) = 64 + 16 + 8 = 216$. And the base 4 equivalent of 216 is $192 + 16 + 8 = 4^3(3) + 4^2(1) + 4^1(2) + 4^0(0) =$ **3120_4**.

3. For any even integer n from 1 to 100, n^2 also is an even integer, which means $n^2 - 15$ will be an odd integer since the difference between an odd integer and an even integer always is odd. It follows, then, that $(n^2 - 15)/2$ will never yield an integer result.

For any odd integer n from 1 to 100, n^2 also is an odd integer, which means that $n^2 - 15$ will be an even integer since the difference between two odd integers always is even. At this point we can't say for certain that $(n^2 - 15)/2$ is even because the quotient of an even integer and 2 is not always even. Since n^2 is odd, it can be written as one less than two times some odd integer. If $(n^2 - 15)/2$ is even it can be written as two times some integer k. That means $(2 \times \text{odd} - 1 - 15)/2 = 2k$. Simplifying, we get $(2 \times \text{odd} - 16)/2 = 2k \rightarrow \text{odd} - 16 = 2k$. But this is a contradiction since the difference between an odd integer and an even integer always is odd and $2k$ always is even.

Therefore, **0** integers n exist from 1 to 100 such that $(n^2 - 15)/2$ is even.

The Problem is solved in the MATHCOUNTS® Mini video.

Follow-up Problems

4. In base 10, $AB_9 = 9^1(A) + 9^0(B) = 9A + B$. In base 10, $BA_7 = 7^1(B) + 7^0(A) = 7B + A$. We are told that $AB_9 = BA_7$, so we can write $9A + B = 7B + A \rightarrow 8A = 6B \rightarrow A/B = 6/8 = 3/4$. Now $A/B = 3/4$ and the only values of A and B that work are A = 3 and B = 4. (Note that if A = 6, then B = 8. But the digit 8 does not exist in base 7, since base 7 numbers are expressed using the digits 0 through 6). Thus, in base 10, $AB_9 = 9(3) + 4 = 27 + 4 =$ **31**, and $BA_7 = 7(4) + 3 = 28 + 3 =$ **31**.

5. One way to tackle this particular problem is to determine the following products.

$11^2 = 121$

$$11^4 = 121^2: \quad \begin{array}{r} 121 \\ \times 121 \\ \hline 121 \\ 2420 \\ \underline{12100} \\ 14641 \end{array}$$

$$11^8 = 14641^2: \quad \begin{array}{r} 14641 \\ \times 14641 \\ \hline 14641 \\ 585640 \\ \vdots \\ \hline \ldots 81 \end{array}$$

Since we are only interested in the tens digit of 11^8, we can stop here with the knowledge that the tens digit of the decimal form of 11^8 is **8**.

Another way to approach the problem is by using binomial expansion. Since 11 = (10 + 1), it follows that $11^8 = (10 + 1)^8$. Those familiar with this concept will recall that Pascal's Triangle is useful for determining the coefficients when $(x + y)^n$ is expanded. Let's consider the expression $(x + 1)^8$, where $x = 10$. When expanded, it will have the form $Ax^8 + Bx^7 + Cx^6 + Dx^5 + Ex^4 + Fx^3 + Gx^2 + Hx + 1$. We look at the numbers in Row 8 of Pascal's Triangle to get the coefficients. These numbers are 1, 8, 28, 56, 70, 56, 28, 8 and 1. Substituting these values into the expression, we have $x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1$. Now, substituting for 10 for x, yields $10^8 + 8(10)^7 + 28(10)^6 + 56(10)^5 + 70(10)^4 + 56(10)^3 + 28(10)^2 + 8(10) + 1$. We see now that there are **8** tens in the decimal form of 11^8.

6. Let's use the rule to generate a few sequences and see if a pattern emerges.

$F = 1$: 1, 2, 4, 8, 16, ...	1 is not a sweet number.
$F = 2$: 2, 4, 8, 16, ...	2 is not a sweet number.
$F = 3$: 3, 6, 12, 24, 48, 36, 24, 48, 36, ...	<u>3 is a sweet number.</u>
$F = 4$: 4, 8, 16, ...	4 is not a sweet number.
$F = 5$: 5, 10, 20, 40, 28, 16, ...	5 is not a sweet number.
$F = 6$: 6, 12, 24, 48, 36, 24, 48, 36, ...	<u>6 is a sweet number.</u>

So far, the only sweet numbers we've encountered are multiples of three. But does that mean that only multiples of three are sweet numbers? And if so, does that mean that every multiple of three is a sweet number? Let's generate a few more sequences to see.

$F = 7$: 7, 14, 28, 16, ...	7 is not a sweet number.
$F = 8$: 8, 16, ...	8 is not a sweet number.
$F = 9$: 9, 18, 36, 24, 48, 36, 24, 48, ...	<u>9 is a sweet number.</u>

Again, we see that only multiples of three are sweet numbers, and every multiple of three we've tried is a sweet number. Let's try to determine why this is true.

Everytime we double a multiple of three we'll get another multiple of three. And since 16 is not a multiple of three, it will never appear as a result of doubling a term in any sequence that begins with a multiple of three. Everytime we subtract 12 from a multiple of three, the result is another multiple of three because 12 itself is a multiple of three. So, once again, since 16 is not a multiple of three, it will never appear as a result of subtracting 12 from a term in any sequence that begins with a multiple of three. So of the whole numbers from 1 through 50, it follows that 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45 and 48 are the **16** sweet numbers.