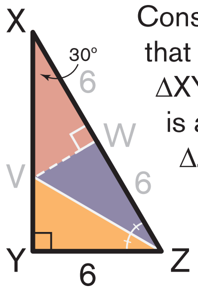
MATHCOUNTS® Minis December 2015 Activity Solutions

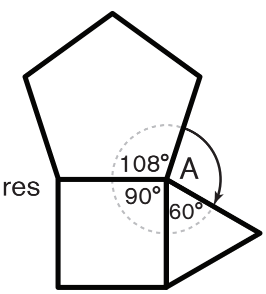
Warm-Up!

1. You may recognize that ΔXYZ is a 30-60-90 right triangle. Recall this is a type of right triangle with special properties that can be used to determine the lengths of the missing sides. in 30-60-90 right triangles the length of the shorter leg is always half the length of the hypotenuse The length of the longer leg is always $\sqrt{3}$ times the length of the shorter leg. Based on this, we have YZ = 6, XZ = **12** and XY = $\mathbf{6\sqrt{3}}$. If, however, you are not familiar with this type of right triangle or its properties, we offer the following alternate solution.



Construct segment ZV to bisect angle Z of right triangle XYZ and to intersect leg XY. Notice that isosceles triangle XVZ is created and right triangle ZYV is created similar to the original ΔXYZ. In isosceles triangle XVZ, construct altitude VW perpendicular to base XZ, which is also the hypotenuse of ΔXYZ. Two congruent right triangles are created, ΔXWV and ΔZWV, each of which is also similar to the original ΔXYZ. Since $\Delta XWV \cong \Delta ZWV$ and $\Delta XYZ \sim \Delta ZYV \sim \Delta XWV \sim \Delta ZWV$, it follows that YZ = WX = WZ = 6 and XZ = 6 + 6 = **12**. Now, we can use the Pythagorean Theorem to determine that $XY = \sqrt{(12^2 - 6^2)} = \sqrt{(144 - 36)} = \sqrt{108} = \sqrt{(36 \times 3)} = \mathbf{6\sqrt{3}}$.

2. Since all of the polygons are regular, we know the measure of the interior angles of each. Each interior angle of an equilateral triangle measures 60 degrees, and each interior angle of a square measures 90 degrees. For a pentagon, each interior angle measures 180 − (360 ÷ 5) = 108 degrees. As the figure shows, the sum of the measures of an interior angle of each polygon and angle A must be 360 degrees. Therefore, the measure of angle A is 360 − (108 + 90 + 60) = 360 − 258 = **102** degrees.



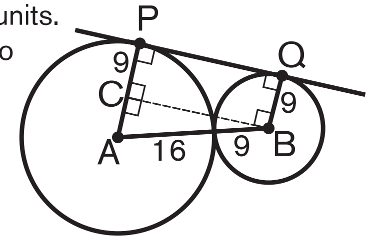
3. Since $\Delta SPQ \sim \Delta STU$ (Angle-Angle), the ratios of corresponding sides of the triangles are congruent. We are told that SP = 2PT → ½(SP) = PT. Since ST = SP + PT, we can write ST = SP + ½(SP) → ST = 3⁄2(SP) → SP/ST = 3/2. Since the ratio of corresponding sides of the triangles is 2/3, the ratio of the area of ΔSPQ to the area of ΔSTU is $2^2/3^2 = 4/9$. We also are told that the area of ΔSTU is 45 cm^2. So, it follows that the area of ΔSPQ = 4⁄9(45) = **20** cm^2.

4. There are many rectangles that can be drawn in triangle ABC with one vertex at A and each of the other three vertices on a side of the triangle. While each of these rectangles has exactly the same perimeter, they do not all have the same area. We are asked to determine the largest possible area of one the of these rectangles. Consider, for a moment, the five rectangles with integer dimensions 1 × 5, 2 × 4, 3 × 3, 4 × 2, 5 × 1. The perimeter of each of these rectangles is 12 units, but they have areas 5, 8, 9, 8 and 5 $units^2$, respectively. Notice that the closer in measure the length and width are, the greater the area of the rectangle, and that the rectangle in which the length and width are equal has the greatest area. So, the rectangle with the largest area has one vertex at A and the other three vertices at the midpoints of each side. It is a square with side length 3 units and area **9** $units^2$.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. In the figure shown here, we have added the segment from B that is perpendicular to radius AP. This segment completes rectangle BCPQ, and now BQ = PC, so PC = 9 units. Radius AP is 16 units, so AC = 16 − 9 = 7 units. When we connect the two centers of the externally tangent circles, we get AB = 16 + 9 = 25 units. Now, using the Pythagorean Theorem with right triangle ABC, we have $25^2 = 7^2 + (BC)^2 \rightarrow 625 = 49 + (BC)^2 \rightarrow (BC)^2 = 576 \rightarrow BC = 24$ units. Because of rectangle BCPQ, we now know PQ = **24** units, too.



6. Since PQ = PS + SQ and we are told that PQ = 3, we have 3 = PS + SQ → PS = 3 − SQ. For similar triangles PQR and PST, we can write the following proportion: 3/(3 − SQ) = 4/ST. Because QSTU is a square, it follows that SQ = QU = UT = ST. Substituting, we get 3/(3 − ST) = 4/ST. Cross-multiplying and solving, we see that 3(ST) = 4(3 − ST) → 3(ST) = 12 − 4(ST) → 7(ST) = 12 → ST = **12/7** units.

7. We are asked to find the measure of ∠BAD and are told that AB||CD. Those familiar with the relationship between two interior angles on the same side of a transversal will recognize that angles BAD and ADC have such a relationship, and, therefore, these angles are supplementary. We have the equation $m\angle BAD + m\angle ADC = 180$. Since we know that $m\angle ADC = 50$, we can substitute and solve to get $m\angle BAD + 50 = 180 \rightarrow m\angle BAD =$ **130** degrees.

For those who are not familiar with the relationship between two interior angles on the same side of a transversal, we can solve this problem another way. We see that the measure of ∠BAD equals the sum of the measures of angles BAC and CAD. We are told that AB||CD, so by the properties of alternate interior angles, it follows that $m\angle BAC = m\angle DCA$. The sum of the measures of the angles of ΔACD must be 180 degrees, so we can write $m\angle ADC + m\angle DCA + m\angle CAD = 180 \rightarrow m\angle CAD = 180 - m\angle ADC - m\angle DCA$. Since we know that $m\angle ADC = 50$ degrees, we can substitute for $m\angle ADC$ and $m\angle DCA$ to get $m\angle CAD = 180 - 50 - m\angle BAC \rightarrow m\angle CAD = 130 - m\angle BAC$. Now we have $m\angle BAD = m\angle BAC + m\angle CAD = m\angle BAC + 130 - m\angle BAC =$ **130** degrees.

8. In the figure, radius OB divides triangle AOE into isosceles triangles ABO and BOE. We have marked their base angles z and x, respectively. We know that $2z + y = 180$ degrees and $x + y = 180$ degrees. This implies that $x = 2z$ and $2x = 4z$. It also is true that $2x + w = 180$ degrees. Substituting $4z$ for $2x$, we get $4z + w = 180$ degrees. Since angle COE has measure 180 – 60 = 120 degrees, it follows that $z + w = 120$ degrees. Subtracting, we get $(4z + w) - (z + w) = 180 - 120$, so $3z = 60$ degrees and $z = 20$ degrees. Therefore, the measure of angle A is **20** degrees.