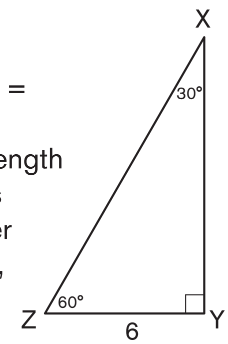
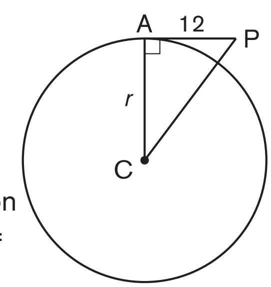
MATHCOUNTS® Minis **April 2018 Activity Solutions**

Warm-Up!

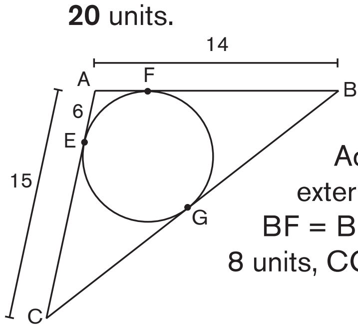


1. Since the $m\angle X = 30°$ and $m\angle Y = 90°$, it follows that the $m\angle Z = 180 - (90 + 30) = 180 - 120 = 60°$. Therefore, $\triangle XYZ$ is a 30-60-90 right triangle. In a 30-60-90 right triangle, the length of the longer leg, which is opposite the 60° angle, is $\sqrt{3}$ times the length of the shorter leg, which is opposite the 30° angle, and the length of the hypotenuse is twice that of the shorter leg. As the figure shows, in triangle XYZ, side YZ is the shorter leg, side XY is the longer leg and side XZ is the hypotenuse. We are told YZ = 6 units, so it follows that XY = $\mathbf{6\sqrt{3}}$ units and XZ = **12** units.

2. The area of rectangle ABCD is $27(11) = 297$ units^2. If we subtract the area of the triangular region that is removed from the area of rectangle ABCD, the result is the area of pentagon ABEFD. Now $CF = CD - FD = 27 - 15 = 12$ units, and $EC = BC - BE = 11 - 6 = 5$ units. Thus the area of $\triangle CEF$ is $(1/2)(12)(5) = 30$ units^2. That means the area of pentagon ABEFD is $297 - 30 =$ **267** units^2.



3. Since segment AP is tangent to the circle at A, segment PA will be perpendicular to radius AC. Because the area of the circle is 256π units^2, we can write the following equation and solve for r: $256\pi = \pi r^2 \rightarrow 256 = r^2 \rightarrow r = 16$ units. Using the Pythagorean Theorem with right triangle APC, we now can write the following equation and solve for PC: $(PC)^2 = 12^2 + 16^2 \rightarrow (PC)^2 = 144 + 256 \rightarrow (PC)^2 = 400 \rightarrow PC =$ **20** units.

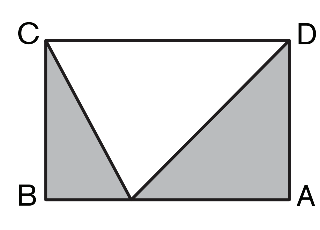


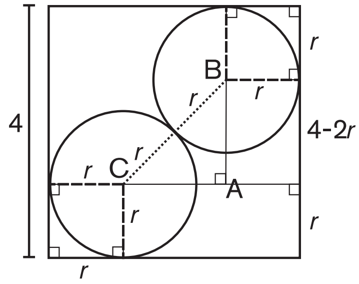
4. Since $AC = AE + EC$, we can determine $EC = 15 - 6 = 9$ units. Additionally, from a point outside of the circle, the two segments from that exterior point to the two different points of tangency are equal. Thus, AE = AF, BF = BG and CG = CE. It follows that AF = 6 units, $BF = 14 - 6 = 8$ units, BG = 8 units, CG = 9 units, and finally, $CB = 9 + 8 =$ **17** units.

The Problems are solved in the MATHCOUNTS® Mini video.

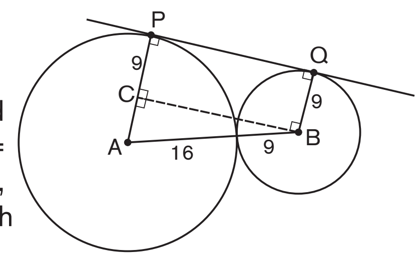
Follow-up Problems

5. The figure appears to be a rectangle from which side CD has been removed and two of the three interior triangles have been shaded, as shown. If we determine the area of rectangle ABCD and subtract from it the area of the unshaded triangle, the result will be the area of the shaded region. The area of the rectangle is $(8)(12) = 96$ cm^2. The area of the unshaded triangle is $(1/2)(12)(8) = (6)(8) = 48$ cm^2. That means the area of the shaded region is $96 - 48 =$ **48** cm^2.





6. In the figure shown here, we have indicated the different radii and right angles that are known from the information given. Additionally, since each side of the square is 4 units, we can see how the right side is divided into segments of lengths r units, r units and $4 - 2r$ units. Thus, AB = $4 - 2r$. Similarly, AC = $4 - 2r$. Seeing that BC = $2r$ and using the Pythagorean Theorem with right triangle ABC, we can write the following equation and solve for r: $(2r)^2 = (4 - 2r)^2 + (4 - 2r)^2 \to 4r^2 = 16 - 16r + 4r^2 + 16 - 16r + 4r^2 \to 0 = 4r^2 - 32r + 32 \to 0 = r^2 - 8r + 8$. Using the Quadratic Formula with $a = 1$, $b = -8$ and $c = 8$, we get $r = \dfrac{8 \pm \sqrt{(-8)^2 - 4(1)(8)}}{2(1)} = \dfrac{8 \pm \sqrt{64 - 32}}{2} = \dfrac{8 \pm \sqrt{32}}{2} = \dfrac{8 \pm 4\sqrt{2}}{2} = 4 \pm 2\sqrt{2}$.
Since $4 + 2\sqrt{2}$ is too large (it's greater than the side of the square), the radius is $\mathbf{4 - 2\sqrt{2}}$ units.



7. In the figure shown here, we have added the segment from B that is perpendicular to radius AP. This segment completes rectangle BCPQ, and now BQ = PC, so PC = 9 units. Radius AP is 16 units, so AC = 16 − 9 = 7 units. When we connect the two centers of the externally tangent circles, we get AB = 16 + 9 = 25 units. Now, using the Pythagorean Theorem with right triangle ABC, we have $25^2 = 7^2 + (BC)^2 \to 625 = 49 + (BC)^2 \to (BC)^2 = 576 \to BC = 24$ units. Because of rectangle BCPQ, we now know PQ = **24** units, too.

8. Let's extend segments AD and BC until they intersect at point E, as shown. Notice that $m\angle EBA = 180 - 120 = 60°$, and $m\angle BAE = 180 - 90 = 90°$. That means the $m\angle E = 30°$, and $\triangle ABE$ is a 30-60-90 right triangle. We know that AB = 3, so using the properties of 30-60-90 right triangles, we see that EB = 2 × 3 = 6. Now consider right triangle CDE with $m\angle C = 90°$ and $m\angle E = 30°$. It follows that $m\angle D = 60°$ making $\triangle CDE$ a 30-60-90 right triangle. The length of the longer leg is EC = EB + BC = 6 + 4 = 10. Segment CD is the shorter leg of $\triangle CDE$. Therefore, according to the properties of 30-60-90 right triangles, we have CD = $\dfrac{EC}{\sqrt{3}} = \dfrac{10}{\sqrt{3}} = \mathbf{\dfrac{10\sqrt{3}}{3}}$.