April 2017 Activity Solutions

Warm-Up!

1. (a) One million is 1,000,000. Granted, we're not supposed to be writing anything, but just looking at this, I can see that $1{,}000{,}000 = 1000^2$, so the square of **999** (or 999^2) would be the largest square less than one million.

   (b) This means we need the smallest positive three-digit integer that is 1 more than a multiple of 7. Let's build our smallest positive three-digit multiple of 7. It must be of the form 10_. Dividing 10 by 7 leaves a remainder of 3, and then making the units digit a 5 would make the situation such that 7 now evenly divides into 35. So 105 is the smallest positive three-digit multiple of 7 and **106** is the answer to the original question. (You may have gone a different route... since 77 is a multiple of 7, if we continue to add 7 or multiples of 7 we can find the number we're looking for. Adding 21 to 77 gives us 98, so 99 would give us a remainder of 1, but isn't a three-digit number; and then adding 7 more we get 106, so this is the smallest positive three-digit integer that is one more than a multiple of 7.)

   (c) This is similar to the previous question. I know 999 is a multiple of 9, and that's pretty close to a four-digit number. In fact, if we add 5, we get 1004, and we know that **1004** will leave a remainder of 5 when it's divided by 9, so this is our answer.

2. There are 8 letters in the repeating pattern. We know that the letters in the 8th, 16th, 24th, ... 2000th, 2008th positions all will be E, the last letter in the repeating pattern. That means that the letter in the 2009th position will be M, the first letter in the repeating pattern. And the letter in the 2010th position will be **A**, the second letter in the repeating pattern.

3. Let's start with 993 because we know it's the largest integer that satisfies the second part of the question. However, is it 7 greater than a multiple of 9? $993 - 7 = 986$, and this is not a multiple of 9. Let's try $993 - 11 = 982$ (which we know still satisfies the second requirement). However, $982 - 7 = 975$ is not a multiple of 9.

   Let's try $982 - 11 = 971 \rightarrow 971 - 7 = 964 \rightarrow$ not a multiple of 9.
   Let's try $971 - 11 = 960 \rightarrow 960 - 7 = 953 \rightarrow$ not a multiple of 9.
   Let's try $960 - 11 = 949 \rightarrow 949 - 7 = 942 \rightarrow$ not a multiple of 9.
   Let's try $949 - 11 = 938 \rightarrow 938 - 7 = 931 \rightarrow$ not a multiple of 9.
   Let's try $938 - 11 = 927 \rightarrow 927 - 7 = 920 \rightarrow$ not a multiple of 9.
   Let's try $927 - 11 = 916 \rightarrow 916 - 7 = 909 \rightarrow$ a multiple of 9.

   So our answer is **916**.

4. The powers of 3 have a repeating pattern when divided by 5.

   $3^1 = 3 \rightarrow 3 \div 5 = 0$ remainder 3
   $3^2 = 9 \rightarrow 9 \div 5 = 1$ remainder 4
   $3^3 = 27 \rightarrow 27 \div 5 = 5$ remainder 2
   $3^4 = 81 \rightarrow 81 \div 5 = 16$ remainder 1
   $3^5 = 243 \rightarrow 243 \div 5 = 48$ remainder 3

   Now we see that the four-number pattern 3, 4, 2, 1 repeats for the remainders when powers of 3 are divided by 5. Since 2008 is a multiple of 4, it follows that 3^{2008} will have the same remainder when divided by 5 as 3^4 does. Therefore, the remainder when 3^{2008} is divided by 5 will be **1**.

The Problems are solved in the **MATHCOUNTS**® *Mini* video.

Follow-up Problems

5. We are looking for an integer that leaves a remainder of 3 when divided by 5 and a remainder of 7 when divided by 9 and by 13. We see that dividing by either 9 or 13 will result in a remainder of 7. Based on this information, our number could be 7, 7 + (9)(13), 7 + (9)(13) + (9)(13), etc. The value of (9)(13) is 117. When 117 is divided by 5, the remainder is 2.

	7 +	(9)(13) +	(9)(13) +	(9)(13) +	(9)(13) +	...
Remainder when divided by 5	2	2	2	2	2	...

We see that, when divided by 5,
7 + (9)(13) has a remainder of 2 + 2 = 4;
7 + (9)(13) + (9)(13) has a remainder of 2 + 2 + 2 = 6, or 1 since 6 ÷ 5 = 1 R1;
7 + (9)(13) + (9)(13) + (9)(13) has a remainder of 2 + 2 + 2 + 2 = 8, or 3 since 8 ÷ 5 = 1 R3.
This must be our desired value, 7 + (9)(13) + (9)(13) + (9)(13) = 7 + 117 + 117 + 117 = **358**.

6. Whoa! This seems way more complicated. Let's look at our scenario in a table. Though it seems to have many more requirements, notice that some of them are repetitive. For instance, if a number leaves a remainder of 3 when divided by 4, it will also leave a remainder of 1 when divided by 2, so we really don't need the first row of our table. Similarly, a number that leaves a remainder of 5 when divided by 6 will also leave a remainder of 2 when divided by 3, so the second row of our table is not necessary either. Take a look at the scenarios of dividing by 4 and dividing by 6. These divisors have some common factors. The numbers 3, 7, 11, 15, ... satisfy the conditions for dividing by 4. The numbers 5, 11, 17, 23, ... satisfy the conditions for dividing by 6. The number 11 is in both lists, so satisfies both conditions. So will 11 plus any multiple of 12 (which is the least common multiple of 4 and 6. The requirements for dividing by 4 and 6 can be rephrased as finding a number that has a remainder of 11 when divided by 12. So now we have the scenario shown to the left. This seems a bit more manageable. The first row tells us our number will have a units digit of 4 or 9. Using the second row, we see 6, 13, 20, 27, 34, ... work. Using the third row, we see 11, 23, 35, 47, work. There isn't any overlap yet, so let's keep going:

Divide by	Remainder
2	1
3	2
4	3
5	4
6	5
7	6

Divide by	Remainder
5	4
7	6
12	11

6, 13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90, ...
11, 23, 35, 47, 59, 71, 83, 95, 107, 119, 131, 143, ...

So now we know 83 works for the last two rows, and so will 83 + (7)(12), 83 + (7)(12) + (7)(12), 83 + (7)(12) + (7)(12) + (7)(12), etc. We need to find which of these have a units digit of 4 or 9. Since (7)(12) = 84, we can simplify this list of possibilities to 83, 167, 251, 335, **419**, ...

Want to see an easier solution? In the original table we can see that if our desired number was just 1 more than it is, it would be exactly divisible by 2, 3, 4, 5, 6 and 7. This means our number is 1 less than the least common multiple of 2, 3, 4, 5, 6 and 7. Notice that 12 is the least common multiple of 2, 3, 4 and 6. So our number is (5)(7)(12) − 1 = 420 − 1 = **419**.

7. Jan will step on steps 13, 16, 19, 22, 25, 28, 31, ..., 130. Jen will step on steps 3, 7, 11, 15, 19, 23, 27, 31, ..., 139. Since Jan is stepping on every 3rd step and Jen is stepping on every 4th step, it makes sense that every 3 × 4 = 12th step will be stepped on by both Jan and Jen. Notice that Jan and Jen both step on steps 19 and 31, and that 31 − 19 = 12. We can continue the sequence to see that 19, 31, 43, 55, 67, 79, 91, 103, 115 and 127 are the **10** steps that are stepped on by both Jan and Jen.

This can also be determined without listing the entire sequence of steps. We know that the range of steps on which Jan and Jen both step is from 19 to 130. Since 130 − 19 = 111 and 111 ÷ 12 = 9.25, we can determine that, after 19, another 9 steps are stepped on by both Jan and Jen, for a total of **10** steps.

8. Let's, first, try a few different combinations of positive odd integers x, y and z to see what the remainder is when the sum of their squares is divided by 4. The table below shows four such scenarios.

x	y	z	$x^2 + y^2 + z^2$	$\frac{x^2 + y^2 + z^2}{4}$
1	1	1	3	0 R.3
1	1	3	11	2 R.3
1	3	3	19	4 R.3
1	3	5	35	8 R.3

It appears that for all positive odd integers x, y and z, when the sum of their squares is divided by 4, the remainder will always be 3. Let's see if we can prove this algebraically. For nonnegative integers u, v and w, let $x = 2u + 1$, $y = 2v + 1$ and $z = 2w + 1$, since x, y and z are odd integers. Then the sum of the squares of x, y and z is $x^2 + y^2 + z^2 = (2u + 1)^2 + (2v + 1)^2 + (2w + 1)^2 = 4u^2 + 4u + 1 + 4v^2 + 4v + 1 + 4w^2 + 4w + 1 = 4(u^2 + u + v^2 + v + w^2 + w) + 3$. This proves that the sum of the squares of all positive odd integers x, y and z will always leave a remainder of **3** when divided by 4.