MATHCOUNTS® Minis April 2016 Activity Solutions

Warm-Up!

1. Let's classify each parallegram based on the number of rows and columns it spans. Then we will examine the various cases.
Case 1: Consider all parallelograms that span 1 row and 1 column. There are 4 such parallelograms.
Case 2: Consider all parallelograms that span 1 row and 2 columns. There are 2 such parallelograms.
Case 3: Consider all parallelograms that span 2 rows and 1 column. There are 2 such parallelograms.
Case 4: Consider all parallelograms that span 2 rows and 2 columns. There is 1 such parallelogram.
These four cases yield a total of 4 + 2 + 2 + 1 = **9** parallelograms.

2. Since one of the numbers will be included in the calculation for both the column and the row, we are looking for two pairs of numbers, selected from 1, 5, 9, 13 and 17, that have the same sum. There are two cases to consider.
Case 1: The pairs 1 + 13 and 9 + 5 both add up to 14. In this case, the center square would be 17, and the row sum would be 1 + 17 + 13 = 9 + 17 + 5 = 31.
Case 2: The pairs 1 + 17 and 5 + 13 both add up to 18. In this case, the center square would be 9, and the row sum would be 1 + 9 + 17 = 5 + 9 + 13 = 27.
Based on these two cases, the greatest possible row sum is **31**.

3. Let's start by maximizing the number of dimes and examine the various cases as we reduce the number of dimes.
Case 1: Using two dimes, Jamie can make 26¢ in 2 ways:
10¢ + 10¢ + 5¢ + 1¢ = 26¢
10¢ + 10¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢
Case 2: Using one dime, Jamie can make 26¢ in 2 ways:
10¢ + 5¢ + 5¢ + 5¢ + 1¢ = 26¢
10¢ + 5¢ + 5¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢
Case 3: Using no dimes, Jamie can make 26¢ in 1 way:
5¢ + 5¢ + 5¢ + 5¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ + 1¢ = 26¢
Based on these three cases, we see that given 2 dimes, 4 nickels and 8 pennies, Jamie can make 26¢ in 2 + 2 + 1 = **5** ways

4. There are three triples that add up to 6. Let's examine these three cases:
Case 1: The triple 2, 2 and 2 can be ordered in 1 way.
Case 2: The triple 1, 1 and 4 can be ordered in 3!/2 = 3 ways.
Case 3: The triple 1, 2 and 3 can be ordered in 3! = 6 ways.
These three cases yield a total of 1 + 3 + 6 = **10** ordered triples.

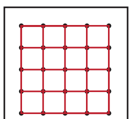
The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

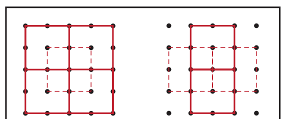
5. There are two ways for a log to get from pond A to pond B: A --> K --> B or A --> J --> B.
Case 1: The probability that a log takes the route A --> K --> B is (1/3) × (1/2) = 1/6.
Case 2: The probability that a log takes the route A --> J --> B is (1/3) × (1/3) = 1/9.
Based on these two case, the probability of a log going from pond A to pond B is 1/6 + 1/9 = (3 + 2)/18 = **5/18**.

6. This problem is tricky. We can't arbitrarily select two horizontal lines and two vertical lines to form a square. Once you select two horizontal lines the vertical lines have to be selected in such a way that the horizontal length is the same as the vertical length. Another consideration is that squares can be formed in the grid by combining a pair of horizontal lines with a pair of vertical lines and by combining a pair of diagonal lines with a positive slope with a pair of diagonal lines with a negative slope. Let's first determine how many squares can be created using only horizontal and vertical lines. We'll classify each square by its dimensions and consider the various cases, starting with the number of 1×1 squares.

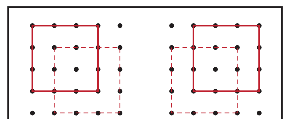
Case 1: 1×1 squares


1 × 1
16 squares

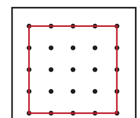
Case 2: 2×2 squares


2 × 2
9 squares

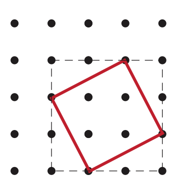
Case 3: 3×3 squares


3 × 3
4 squares

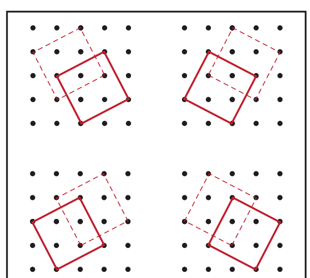
Case 4: 4×4 squares


4 × 4
1 square

Now we will consider the possible cases of diagonal squares. The figure below on the left shows a diagonal square (solid line) inside a 3×3 region (dotted line) of the grid. To determine the side lengths of the square we can use the Pythagorean Theorem. Notice the right triangles in each corner of the 3×3 region. Each of these triangles has a short leg with length 1 unit, a long leg with length 2 units and a hypotenuse that is a side of the diagonal square. We know the length of the hypotenuse is $c = \sqrt{(a^2 + b^2)} \rightarrow c = \sqrt{(1^2 + 2^2)} \rightarrow c = \sqrt{5}$. We can form 7 more squares of this size on our grid, as shown below on the right.

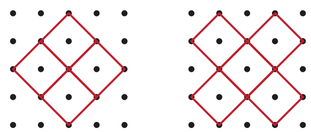


Case 5: $\sqrt{5} \times \sqrt{5}$ squares


$\sqrt{5} \times \sqrt{5}$
8 squares

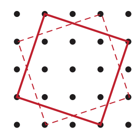
But we need to determine all the possible side lengths for these diagonal squares to ensure that we don't miss any. To find all the possible side lengths, consider that the side of the square will be the hypotenuse of a right triangle and we can write $c = \sqrt{(a^2 + b^2)}$, where c is the length of the hypotenuse and a and b are the lengths of the shorter and longer legs of the triangle, respectively. If $a = 1$, we can form right triangles such that $b = 1$, $b = 2$ or $b = 3$. It doesn't work if $b = 4$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(1^2 + 1^2)} = \sqrt{2}$, $c = \sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $c = \sqrt{(1^2 + 3^2)} = \sqrt{10}$. If $a = 2$, we can form right triangles such that $b = 1$ or $b = 2$. Again, it doesn't work if $b = 3$ because the square would extend beyond our grid. That gives us squares with side lengths $c = \sqrt{(2^2 + 1^2)} = \sqrt{5}$ and $c = \sqrt{(2^2 + 2^2)} = 2\sqrt{2}$. Below are the squares that can be formed with sides of lengths $\sqrt{2}$ units, $\sqrt{10}$ units and $2\sqrt{2}$ units (the 8 squares with sides of length $\sqrt{5}$ units were shown previously).

Case 6: $\sqrt{2}\times\sqrt{2}$ squares



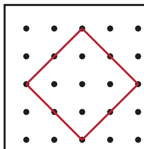
$\sqrt{2} \times \sqrt{2}$
9 squares

Case 7: $\sqrt{10}\times\sqrt{10}$ squares



$\sqrt{10} \times \sqrt{10}$
2 squares

Case 8: $2\sqrt{2}\times 2\sqrt{2}$ squares



$2\sqrt{2} \times 2\sqrt{2}$
1 square

Finally, that brings the total number of squares to 16 + 9 + 4 + 1 + 8 + 9 + 2 + 1 = **50** squares.

7. Let’s start with the case in which the sum includes five 1. Then we'll reduce the number of 1s while listing the sums for each case.
Case 1: In the case of five 1s, there is only $\underline{1}$ sum: 13 + 1 + 1 + 1 + 1 + 1.
Case 2: In the case of four 1s, the other two odd numbers must add up to 14. There are three possibilities: 11 + 3, 9 + 5 and 7 + 7. This case yields $\underline{3}$ sums.
Case 3: In the case of three 1s, the other three odd numbers must add up to 15. There are three possibilities: 9 + 3 + 3, 7 + 5 + 3 and 5 + 5 + 5. This case yields $\underline{3}$ sums.
Case 4: In the case of two 1s, the other four odd numbers must add up to 16. There are two possibilities: 7 + 3 + 3 + 3 and 5 + 5 + 3 + 3. This case yields $\underline{2}$ sums.
Case 5: In the case of one 1, the other five odd numbers must add up to 17. There is one possibility: 5 + 3 + 3 + 3 + 3. This case yields $\underline{1}$ sum.
Case 6: In the case of no 1s, there is one possibility with six odd numbers that add up to 18: 3 + 3 + 3 + 3 + 3 + 3. This case yields $\underline{1}$ sum.
These six cases yield a total of 1 + 3 + 3 + 2 + 1 + 1 = **11** sums.

8. Let's start by determining which perfect squares can obtained by multiplying two different numbers from 1 to 16, inclusive. Since 15 × 16 = 240, the perfect squares in question will all be less than 240. The perfect squares less than 240 are 1, 4, 9, 16, 25, 36 ,49, 64, 81, 100, 121, 144, 169, 196 and 225. The perfect squares that can not be obtained by multiplying two different numbers from 1 to 16, inclusive are 1, 25, 49, 81, 100, 121, 169, 196 and 225. We can, however, obtain the following products, which are perfect squares:

4 = 1 × 4	9 = 1 × 9	36 = 3 × 12 = 4 × 9
16 = 1 × 16 = 2 × 8	64 = 4 × 16	144 = 9 × 16

To avoid drawing two numbers whose product is a perfect square, Jillian must not draw the numbers 1, 2, 3, 4, 8, 9, 12 and 16. So the first eight numbers Jillian can draw without a pair whose product is a perfect square are 5, 6, 7, 10, 11, 13, 14, 15. There are 2 ways for her to draw the ninth and tenth numbers without a pair whose product is a perfect square:
Case 1: If she draws 2 and 3, that leaves the numbers 1, 4, 8, 9, 12 and 16.
Case 2: If she draws 8 and 12, that leaves the numbers 1, 2, 3, 4, 9 and 16.
In either case, if Jillian next draws the 1, 4, 9 or 16, she will have drawn a total of 11 numbers without a pair whose product is a perfect square. Any number drawn after that will result in a pair of numbers whose product is a perfect square, so the maximum number of slips that Jillian can draw is **11** slips.