April 2014 Activity Solutions

Warm-Up!

1. (a) $(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = \mathbf{x^2 + 2x + 1}$
   (b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = \mathbf{x^2 + 4x + 4}$
   (c) $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = \mathbf{x^2 + 6x + 9}$

In general, we see that $(x + y)^2 = x^2 + 2xy + y^2$ for any real numbers x and y.

2. (a) $(x - 1)(x + 1) = (x - 1)(x) + (x - 1)(1) = x^2 - x + x - 1 = \mathbf{x^2 - 1}$.
   (b) $(x - 2)(x + 2) = (x - 2)(x) + (x - 2)(2) = x^2 - 2x + 2x - 4 = \mathbf{x^2 - 4}$.
   (c) $(x - y)(x + y) = (x - y)(x) + (x - y)(y) = x^2 - xy + xy - y^2 = \mathbf{x^2 - y^2}$.

3. (a) $5^2 - 4^2 = 25 - 16 =$ **9**
   (b) $6^2 - 5^2 = 36 - 25 =$ **11**
   (c) $7^2 - 6^2 = 49 - 36 =$ **13**
   (d) $8^2 - 7^2 = 64 - 49 =$ **15**
   (e) It appears that the difference of the squares of two consecutive integers is equal to the sum of the two consecutive integers. To see if this is always true, let's do a little algebra. Let $n - 1$ and n be two consecutive integers. What do we get when we simplify $n^2 - (n - 1)^2$? Start by expanding $(n - 1)^2$ to get $n^2 - (n^2 - n - n + 1) = n^2 - n^2 + n + n - 1 = n + n - 1 = n + (n - 1)$, which is the sum of the two consecutive integers. Therefore, it is the case that the difference of the squares of two consecutive integers equals the sum of the two consecutive integers.

4. $(13 \times 47 + 13 \times 21 + 13 \times 12) + (21 \times 47 + 21 \times 21 + 21 \times 12) + (46 \times 47 + 46 \times 21 + 46 \times 12)$
   $= 13(47 + 21 + 12) + 21(47 + 21 + 12) + 46(47 + 21 + 12)$
   $= (13 \times 80) + (21 \times 80) + (46 \times 80)$
   $= 80(13 + 21 + 46)$
   $= 80 \times 80$
   $= 80^2$
   $=$ **6400**.

The Problems are solved in the video.

Follow-up Problems

5. Let $x = 20122011$, and let $y = 20122009$. We are asked to evaluate the expression $20122011^2 - 2(20122011)(20122009) + 20122009^2$, which can be rewritten as $x^2 - 2xy + y^2$. Factoring, we see this is equivalent to $(x - y)^2$. So we have $(x - y)^2 = (20122011 - 20122009)^2 = 2^2 =$ **4**.

6. Since $55{,}556 = 55{,}555 + 1$ we can write $55{,}556^2 = (55{,}555 + 1)^2 = 55{,}555^2 + 55{,}555 + 55{,}555 + 1$. Substituting 3,086,358,025 for $55{,}555^2$, we get $3{,}086{,}358{,}025 + (2)(55{,}555) + 1 = 3{,}086{,}358{,}025 + 111{,}1110 + 1 =$ **3,086,469,136**.

7. Notice that $3^8 - 2^6$ can also be written as $(3^4)^2 - (2^3)^2$. From the video we know that the difference of the squares of two integers is the product of the difference and the sum of the two integers. That means $(3^4)^2 - (2^3)^2 = (3^4 - 2^3)(3^4 + 2^3)$. If we simplify this expression, we get $(81 - 8)(81 + 8) =$ **73×89** as the prime factorization.

8. Since $29{,}999{,}999 = 30{,}000{,}000 - 1$, we can write $29{,}999{,}999^2$ as $(30{,}000{,}000 - 1)^2$. Because $(x - 1)^2 = x^2 - 2x + 1$, it follows that $(30{,}000{,}000 - 1)^2 = (30{,}000{,}000)^2 - 2(30{,}000{,}000) + 1 = 9 \times 10^{14} - 6 \times 10^7 + 1 = 90{,}000{,}000 \times 10^7 - 6 \times 10^7 + 1 = 89{,}999{,}994 \times 10^7 + 1$. Therefore, the sum of the digits is $8 + 6(9) + 4 + 1 = 13 + 54 =$ **67**.

9. Let's first look at a similar expression with smaller numbers and simplify it. For example, consider the following expression:

$$\frac{(5^2 - 3^2)(5^2 - 2^2)(5^2 - 1^2)(5^2 - 0^2)}{(4^2 - 3^2)(4^2 - 2^2)(4^2 - 1^2)(4^2 - 0^2)} = \frac{(5 + 3)(5 - 3)(5 + 2)(5 - 2)(5 + 1)(5 - 1)(5 + 0)(5 - 0)}{(4 + 3)(4 - 3)(4 + 2)(4 - 2)(4 + 1)(4 - 1)(4 + 0)(4 - 0)}$$

$$= \frac{(8)\cancel{(2)}\cancel{(7)}\cancel{(3)}\cancel{(6)}\cancel{(4)}\cancel{(5)}(5)}{\cancel{(7)}(1)\cancel{(6)}\cancel{(2)}\cancel{(5)}\cancel{(3)}\cancel{(4)}(4)}$$

$$= \frac{\overset{2}{\cancel{(8)}}(5)}{(1)\cancel{(4)}}$$

$$= (2)(5)$$

$$= 10$$

Algebraically, we have the following:

$$\frac{(n^2 - (n - 2)^2)(n^2 - (n - 3)^2)(n^2 - (n - 4)^2)\cdots(n^2 - 1^2)(n^2 - 0^2)}{((n - 1)^2 - (n - 2)^2)((n - 1)^2 - (n - 3)^2)((n - 1)^2 - (n - 4)^2)\cdots((n - 1)^2 - 1^2)((n - 1)^2 - 0^2)}$$

$$= \frac{(n + n - 2)(n - n + 2)(n + n - 3)(n - n + 3)(n + n - 4)(n - n + 4)\cdots(n + 1)(n - 1)(n + 0)(n - 0)}{(n - 1 + n - 2)(n - 1 - n + 2)(n - 1 + n - 3)(n - 1 - n + 3)(n - 1 + n - 4)(n - 1 - n + 4)\cdots(n - 1 + 1)(n - 1 - 1)(n - 1 + 0)(n - 1 - 0)}$$

$$= \frac{(2n - 2)\cancel{(2)(2n - 3)(3)(2n - 4)(4)\cdots(n + 1)(n - 1)(n)}(n)}{\cancel{(2n - 3)(1)(2n - 4)(2)(2n - 5)(3)\cdots(n)(n - 2)(n - 1)}(n - 1)}$$

$$= \frac{\overset{2}{\cancel{(2n - 2)}}(n)}{\cancel{(n - 1)}}$$

$$= 2n$$

All of the terms in the numerator, except $2n - 2$ and n, will cancel all of the terms in the denominator, except $n - 1$. Since $2n - 2 = 2(n - 1)$, the $n - 1$ terms in the numerator and denominator also cancel each other. Thus, we are left with $2n$ as the value of the expression.

Therefore, $\dfrac{(1998^2 - 1996^2)(1998^2 - 1995^2)\cdots(1998^2 - 0^2)}{(1997^2 - 1996^2)(1997^2 - 1995^2)\cdots(1997^2 - 0^2)} = 2(1998) =$ **3996**.