MATHCOUNTS®

2017 State Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2017 MATHCOUNTS® State Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

2017 State Competition Sprint Round

1. The 7 congruent squares enclose a total area of 112 in^2. Therefore, each square encloses 112 in^2/7 = 16 in^2, each edge having length the square root of that 16 in^2 for a length of 4 inches. Of the 7 squares, 5 are contributing 2 edges to the overall perimeter and 2 are contributing 3 edges. This means the perimeter of the whole figure is composed of 5 × 2 + 2 × 3 = 16 edges, each of length 4 inches Therefore, the total perimeter is 16 × 4 in = **64** inches

2. The sum of the first 10 positive integers is 10 × 11/2 = 55. The only perfect square from 55 − 10 to 55 − 1 is 49, so the missing number is 55 − 49 = **6**.

3. $2 = \frac{x+1}{x}$, so 2x = x + 1 and $x = 1$. Therefore, $\frac{x^2+1}{x^2} = \frac{1^2+1}{1^2} = \mathbf{2}$.

4. Bill has earned $\left\lfloor \frac{\$846}{\$50} \right\rfloor = 16$ discounts, each worth \$5. The total discounts are 16 × \$5 = \$80. The final cost of the supplies is \$846 − \$80 = \$**766**.

5. For integers greater than 2000, the least divisible by 3 is 2001, by 2 is 2002, and by 5 is 2005. These three values sum to **6008**.

6. Since 100 % = 1 = 1/1, the horizontal distance and vertical drop are equal, thereby corresponding to a right isosceles triangle, for which the angle is **45** °.

7. Because the diameter of the sphere increases by 0.2 m from 1.2 m, the values for the radius are half that: an increase by 0.1 m from 0.6 m. The relative increase is the amount of increase divided by the starting value, which expressed as a percentage is $\frac{0.1\ \cancel{\text{m}}}{0.6\ \cancel{\text{m}}} \times 100\ \% = \frac{100}{6}\ \% =$ 16.666 … %, which rounds to the nearest tenth of a percent as **16.7** %.

8. As a minimum, 10 baskets are worth 10 × 2 pt = 20 pt. However, she got 26 pt − 20 pt = 6 pt more than that. The “better” baskets are each worth 3 pt − 2 pt = 1 pt extra. Therefore, the number of “better” baskets is $\frac{6\ \cancel{\text{pt}}}{1\ \cancel{\text{pt}}/\text{bsk}} = \mathbf{6}$ bsk.

9. Volume is proportional to the cube of length. The outer cube has edges 4 times as long as the inner cube, so the outer cube encloses a volume 4^3 = 64 times as much volume enclosed by the inner cube, that is, 64 × 5 cm^3 = 320 cm^3. To get the volume of the “gap”, we must subtract the volume enclosed by the inner cube: 320 cm^3 − 5 cm^3 = **315** cm^3.

10. The rate in this problem is the number of jobs completed divided by the time to complete them. Let t be the amount of time Michael takes to do one job, so Michael’s rate is 1 job/t . Andre’s rate is $\frac{(5/6)\text{ job}}{(3/4)t} = \frac{5}{6} \times \frac{4}{3}\ \frac{\text{job}}{t} = \frac{20}{18}\ \frac{\text{job}}{t} = \frac{10}{9}\ \frac{\text{job}}{t}$, which is $\mathbf{\frac{10}{9}}$ times Michael’s rate, in the desired form of a common fraction.

11. Let b_i be the number of grains collected by bee number i. Then $b_2 = 2b_1 + 1$, $b_3 = 2b_2 + 1 = 4b_1 + 3$, $b_4 = 2b_3 + 1 = 8b_1 + 7$, and so on. We see that the coefficient of b_1 goes up by a factor of 2 each time and the added constant always stays 1 less than the coefficient of b_1. We can generalize this into a formula $b_i = 2^{i-1}b_1 + 2^{i-1} - 1$. Therefore, $b_{11} = 2^{10}b_1 + 2^{10} - 1$. If you do not know $2^{10} = 1024$ (which is actually good to know, not just for MATHCOUNTS but because random access memory (RAM) in a computer is often expressed in terms of this relationship), it can be evaluated as $2^{10} = (2^2)^5 = 4^5 = (4\times4)\times(4\times4)\times4 = 16\times16\times4 = 256\times4 = 1024$. Therefore, the number of grains collected by the last be is 1024 × 100 + 1023 = 102,400 + 1 023 = **103,423** grains.

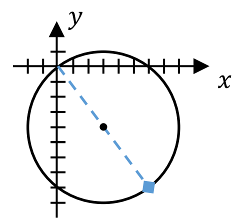
12. There are 9 items with 3 modes, and 3 items do not have modal values. Therefore, at most 6 items have modal values, meaning the frequency of each mode is at most 2. Because there are 3 items given that are definitely not modes, the frequency of a mode must be at least 2. Putting these two statements together results in the modes have frequency 2 and other values have frequency 1. Therefore, a, b, and c each occur twice, and $a + 4$, $1 + b$, and $c - 8$ each occur once, making the mean:
$$17 = \big(2a + 2b + 2c + (a + 4) + (1 + b) + (c - 8)\big)/9 = (3a + 3b + 3c - 3)/9.$$
Now, we want $3a + 3b + 3c$, which is part of the above expression and can be extracted by: $9\times17 = 3a + 3b + 3c - 3$ and $3a + 3b + 3c = 9\times17 + 3 = 153 + 3 = \mathbf{156}$.

13. Let b be the base length of the rectangle and h be the height of the rectangle. We need to find the area bh enclosed by the rectangle. We are given the perimeter $2(b + h) = 40$ in and the length of the diagonal $\sqrt{b^2 + h^2} = 18$ inches. Thus, the semiperimeter is $b + h = 20$ in and the square of the diagonal is $b^2 + h^2 = 324\text{ in}^2$. Squaring the semiperimeter equation yields $b^2 + 2bh + h^2 = 400\text{ in}^2$, which is very similar to the square-of-the-diagonal equation but with a vital extra piece. Taking the square-of-the-semiperimeter equation minus the square-of-the-diagonal equation yields $2bh = 400\text{ in}^2 - 324\text{ in}^2 = 76\text{ in}^2$, and the area is half of this, so $bh = \mathbf{38}\text{ in}^2$. Note that there is no need to find b and h individually to be able to determine their product.

14. Let's work our way across the top row of bricks from left to right. In essence, this involves powers of the binomial $(1 + 1)$, counting the left 1 for a lower-left brick and the right 1 for a lower-right brick; the issue is that not all paths are possible because the wall has ends.
For the leftmost brick, the next row down has only a right brick. There are 3 rows below that, so nominally, $(1 + 1)^3 = 8$ options, except LLL and LLR cannot be done: thus, 6 legal paths.
For the second brick, with 4 rows below it, there are nominally $(1 + 1)^4 = 16$ options, except LLLL and LLLR cannot be done: thus, 14 legal paths. For the third brick, there are the same number of nominal paths as the previous brick, 16, but this time all of them can be done: thus, 16 legal paths. For the fourth brick, there are again 16 nominal paths, but RRRR cannot be done: thus, 15 legal paths. For the rightmost brick, there are again 16 nominal paths, but this time LRRR, RLRR, RRLL, RRLR, RRRL, and RRRR cannot be done: thus, 10 legal paths.
Adding the result of each of these cases yields 6 + 14 + 16 + 15 + 10 = **61** legal paths.

15. The least positive 4-digit value in base 3 is $1000_3 = 3^3 = 27_{10}$; the least positive 3-digit value in base 4 is $100_4 = 4^2 = 16_{10}$. The least value satisfying both is the greater of the two, namely **27**. That decimal value is equal to 1000_3 and 123_4, confirming satisfaction of the stated criteria.

16. $(x-3)^2+(y+4)^2=25$ is the equation of a circle centered at the point $(x,y)=(3,-4)$ with radius √25 = 5. Wanting the maximum value of x^2+y^2 is equivalent to wanting the square of the distance of the farthest point from the origin. Since the origin, (0, 0) is on the circle, the opposite end of the diameter through the origin is the desired point. The diameter is twice the radius, or 10, so the distance of the desired point from the origin is 10, the square of which is **100**. The coordinates of the desired point need not be determined, but the point is at $(x,y)=(6,-8)$.



17. The two least primes are 2 and 3, so q has the form 2^a3^b for some positive integers a and b. The specification that q is the product of a prime and a power in two different ways indicates: $q=2\times 2^{a-1}\times 3^b$ and $q=3\times 2^a\times 3^{b-1}$, where one of $2^{a-1}\times 3^b$ and $2^a\times 3^{b-1}$ is a perfect square and the other is a perfect cube. To be a perfect square, both of the exponents must be even (divisible by 2); to be a perfect cube, both of the exponents must be divisible by 3.
If $2^{a-1}\times 3^b$ is to be the perfect square and $2^a\times 3^{b-1}$ the perfect cube, then a needs to be odd and divisible by 3, and b needs to be even and leave a remainder upon division by 3: The least positive integers satisfying these properties are $a=3$ and $b=4$. Similarly, if we reverse the role of the perfect square and perfect cube, we get $a=4$ and $b=3$. To obtain the least q, we want the lesser exponent on the greater base, so $a=4$ and $b=3$, which yields $q=2^4\times 3^3=2\times 6^3=2\times 216=\mathbf{432}$.

18. Dividing through the second equation by 2, to make both equations equal to 1, yields:
$\frac{1}{2-x}+\frac{4}{y+3}=1=\frac{3}{2-x}+\frac{2}{y+3}$.
Multiplying through the left and right sides by $(2-x)(y+3)$ yields:
$3(y+3)+2(2-x)=1(y+3)+4(2-x)$, which simplifies to:
$2(y+3)=2(2-x)$, so $y+3=2-x$ and all the denominators match in the original equations. Therefore, replacing $y+3$ with $2-x$ in the first original equation yields $5/(2-x)=1$, so $5=2-x$ and $x=-\mathbf{3}$. Substituting this and $y=-x-1=2$ back into the original equations checks out fine with no divisions by 0.

19. Imagining the pole to be like a rolled-up sheet of paper, make a vertical slice down the poll and unroll it into a flat sheet. The width of the sheet is the circumference of the original pole, which is 2π in, because the radius is 1 inches The layout of the snake on this sheet forms a 30°-60°-90° triangle, with the short leg being the circumference of the pole. For one wind around the pole, the snake goes up vertically $\sqrt{3}$ times as far and "uses up" 2 times as much snake. Therefore, 4π in of snake reaches a height of $2\sqrt{3}\pi$ inches Since the snake is 9π in long in total, we can set up the ratio $\frac{9\pi}{4\pi}=\frac{h}{2\sqrt{3}\pi}$. Therefore, the total height is $\left(\frac{9}{4}\right)\left(2\sqrt{3}\pi \text{ inches}\right)=\frac{9}{2}\sqrt{3}\pi$ inches, so $a=9/2$ and $b=3$, and $a+b=\mathbf{15/2}$.

20. Using $4=2^2$ and evaluating the exponents simplifies the numbers to be 2^{81}, 2^{64}, 3^{16}, 2^{54}, and 2^{16}. Now, $2^{16}<3^{16}<4^{16}=(2^2)^{16}=2^{32}$. Putting the values in increasing numeric order, we have: $2^{16}, 3^{16}, 2^{54}, 2^{64}, 2^{81}$. The median is 2^{54}, and the sum of the prime base and the exponent is $2+54=\mathbf{56}$.

21. Two cases must be considered: first the digits 0 through 4, and second the set of 5 subcases where the least digit is any of 1 through 5 (and greatest digit is 1 through 9). The reason for this split is that in the first case the leftmost digit must not be 0 (because we are required to form 5-digit numbers), whereas in the other case any digit may be used in any position.
Case 1 (01234): Any of the 4 nonzero digits may be used in the leftmost slot, and then the remaining 4 digits may fill the remaining slots in any order, resulting in $4 \times 4! = 4 \times 24 = 96$ values.
Case 2 (12345, 23456, 34567, 45678, 56789): In each of the 5 subcases, all 5 digits may fill slots in any order, for 5! = 120 values per subcase. With 5 subcases, there are $5 \times 120 = 600$ values.
Combining the two cases yields $96 + 600 =$ **696** distinct values.

22. The coin and walls correspond to a circle inscribed in a triangle. The radius of a circle inscribed in a triangle is the area of the triangle divided by its semiperimeter. The triangle is a right triangle so its enclosed area is half the product of the leg lengths:
$(1.5 \text{ inches})(2.0 \text{ inches})/2 = 1.5 \text{ in}^2$. The semiperimeter is
$(1.5 \text{ inches} + 2.0 \text{ inches} + 2.5 \text{ inches})/2 = 3$ inches. Therefore, the radius of the inscribed circle is $(1.5 \text{ in}^2)/(3.0 \text{ inches}) = 0.5$ inch However, we are to find the diameter, which is double the radius, so 2×0.5 inch $=$ **1** inch

23. Let A be the digit for the thousands place and B the digit for the hundreds place. (Note that juxtaposition of these letters with each other or with numerals is merely for concatenation of digits to form numerals and is never intended in this problem to indicate implied multiplication.) The number we are interested in is AB28. Dividing 28 by 16 yields a remainder of 12 and dividing 100 by 16 yields a remainder of 4. Therefore, 128 plus any multiple of 400 is divisible by 16, which means B must be odd. That yields 25 options—unnecessarily many, so let's look at the reverse value, 82BA. Dividing 8200 by 16 yields a remainder of 8, so dividing BA by 16 must also yield a remainder of 8, which yields only 6 options for ba: 08; 24; 40; 56; 72; 88. Remember that B must be odd, so only 56 and 72 are candidates. Indeed, **6528** is divisible by 16, but 2728 is not.

24. The radius of a circle inscribed in a triangle is given by the quotient of the area enclosed by the triangle divided by the semiperimeter of the triangle. In this case, we know the length of each side and the triangle is not a right triangle, so let's use Heron's formula to find the area:
$A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the lengths of the three sides, and s is the semiperimeter, $(a+b+c)/2$. But we need to divide area, a, by semiperimeter, s, so let's do that as variables before substituting numbers—the desired radius is $\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.
$s = (9 + 13 + 14)/2 = 18$. Therefore, the desired radius is $\sqrt{9 \times 5 \times 4/18} = \sqrt{\mathbf{10}}$.

25. Filling in two more boxes shows the first box to the right of the #5 box is #20. This means the first layer for the #5 box has $20 - 5 = 15$ boxes. For the next layer each corner box will trigger 3 new boxes (for example, the upper right corner box will trigger a new box above, a new one to the right, and a new corner box), for a net increase of 2 boxes; all other boxes will trigger one new box for no net change. Since there are 4 corners, that means each layer will add 8 boxes more than the previous layer. The first layer adds 15 boxes, so the second through sixth layers will add $23 + 31 + 39 + 47 + 55$ boxes. This is an arithmetic series with sum $(23 + 55) \times 5/2 = 195$, which is the count to add to #20 as the first-layer box. Therefore, the desired box has number $20 + 195 =$ **215**.

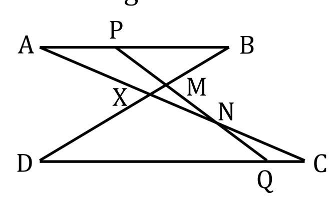
26. The line $\frac{5}{4}x + c$ is parallel to the left edge of the triangle. Therefore, when this line cuts the given triangle, the new triangle is similar to the original triangle. The area of the new triangle is to be 4/(4 + 5) = 4/9 of the original triangle, which means the sides need to be cut down to $\sqrt{4/9} = 2/3$ of the original triangle. The original bottom edge has length 10, so the new bottom edge must have length 20/3 and have right end at $x = 6$, so the left end must be at $x = -2/3$. Thus, the line must pass through $(x, y) = \left(-\frac{2}{3}, 0\right)$, so $0 = \left(\frac{5}{4}\right)\left(-\frac{2}{3}\right) + c = -\frac{5}{6} + c$, and $c = \mathbf{\frac{5}{6}}$.

27. Give 8 of the coins split 2 each among the 4 people to satisfy the restriction. This means there are 12 identical coins to distribute among arbitrarily among 4 people. There are $C(12 + 4 - 1{,}12) = C(15{,}3) = (15\times14\times13)/(1\times2\times3) = \mathbf{455}$ distinct distributions.

28. To avoid division by 0, we must disregard 0 as a possible solution.
First cross-multiply: $x^2 = x^{x-3+4/x}$.
There are in general four cases:
   1. The base is 1.
   2. The base is −1 and the exponents are even.
   3. The base is 0 and the exponents are nonnegative. (We have already rejected this, though.)
   4. The exponents are equal.

With case 1, 1 is indeed a solution.
With case 2, the exponent on the left is 2 and on the right is 0, both of which are even. Therefore, −1 is a solution.
With case 4, $2 = x - 3 + 4/x$, so $0 = x^2 - 5x + 4 = (x - 4)(x - 1)$, making 1 and 4 solutions.
The solution 1 is a repeat (cases 1 and 4) and will be counted only once.
The sum is $-1 + 1 + 4 = \mathbf{4}$.

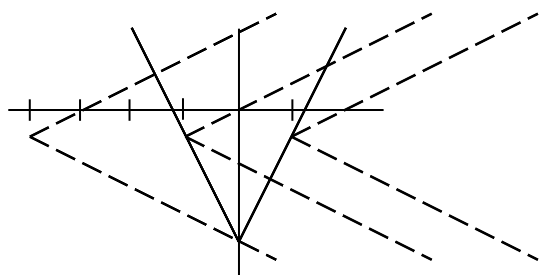
29. The description of the figure is a little hard to keep straight, so let's draw a diagram. For the sake of convenience I put D at the origin and A 3 units up (3 chosen because PQ is split into 3 congruent pieces—ratios are important, not magnitudes, so pick easy-to-work-with magnitudes; also, horizontal shearing has no impact on the result, so choose simplest values). AB:CD = AX:CX = 5:7, so let AB have length 5 and CD length 7. Since the y-component of Q and P is 0 and 3, respectively, to make the sub-segments congruent, the y-component of N and M must be 1 and 2, respectively.



The line through AC has equation $y = 3 - \frac{3}{7}x$, and the line through DB has equation $y = \frac{3}{5}x$.
$1 = 3 - \frac{3}{7}x$ for point N, so the x-component of N is 14/3; $2 = \frac{3}{5}x$ for point M, so the x-component of M is 10/3. Going from N to M decreased the x-component by 4/3; going from M to P will decrease the x-component by another 4/3, so the x-component of P is $10/3 - 4/3 = 2$ and AP has length 2 and PB has length 5 − 2 = 3. Therefore, the ration AB:BP = $\mathbf{2/3}$.

30. The first equation can be rewritten as $y = |2x| - 5$, which is V-shaped with axis of symmetry along the y-axis and vertex at $(0, -5)$ and the second equation as $x = |2y + 2| + a$, which has a V shape opening to the right, axis of symmetry at $y = -1$ and vertex at $(a, -1)$. There are 3 values of a (3 positions of the horizontal V) for which there is an odd number of solutions (x, y):
1. 1 solution when the vertex touches the right side of the vertical V, which is at $(2, -1)$ and $a = 2$.
2. 3 solutions when the vertex touches the left side of the vertical V, which is at $(-2, -1)$ and $a = -2$.
3. 3 solutions when the vertex of the vertical V touches the bottom side of the horizontal V, which is at $(0, -5)$ and $a = -8$.

Therefore, the product of these 3 values of a is $2(-2)(-8) =$ **32**.



2017 State Competition Target Round

1. You might or might not recognize $p^3 + q^3 + r^3 + 3(p + q)(q + r)(p + r) = (p + q + r)^3$.
If you do, the answer is simply $(2 + 4 + 5)^3 = 11^3 = \mathbf{1331}$.
If you do not (and you should not feel badly for not recognizing this), more plugging and grinding is required but it is very straightforward:
$2^3 + 4^3 + 5^3 + 3(2 + 4)(4 + 5)(2 + 5) = 8 + 64 + 125 + 3(6)(9)(7) = 197 + 1134 = \mathbf{1331}$.

2. After 7 days the two dogs have finished $\frac{7\text{ days}}{10\text{ days/bag}} = \frac{7}{10}$ bag. The remaining $\frac{3}{10}$ bag takes Nipper 9 days to finish, so Nipper eats at the rate of $\frac{\frac{3}{10}\text{ bag}}{9\text{ days}} = \frac{1}{30}\frac{\text{bag}}{\text{days}}$, thus requiring 30 days to finish a bag by himself. Therefore, in 7 days, Nipper would have eaten 7 days $\times \frac{1\text{ bag}}{30\text{ days}} = \frac{7}{30}$ bag by himself. In the same 7 days, the two dogs together ate $\frac{7}{10}$ bag. Therefore, Biter ate $\frac{7}{10}$ bag $- \frac{7}{30}$ bag $= \frac{21-7}{30}$ bag $= \frac{14}{30}$ bag, which is 2 times as much as Nipper ate (determining that factor of 2 is easier to determine without reducing 14/30 to 7/15—reducing is critical for final answers but not at all for intermediate calculations). The amount of time a bag of dog food would last is inversely proportional to the amount of food eaten per day. Therefore, the number of days a bag would last Biter is **1/2** of the number of days for Nipper.

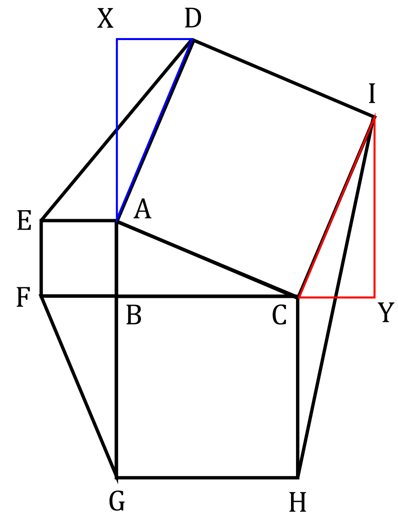
3. Passing exactly one test means to pass one test and fail the other, so the probability is given by:
$P(\text{PS, FM}) + P(\text{FS, PM}) = \frac{3}{4}\left(1 - \frac{4}{5}\right) + \left(1 - \frac{3}{4}\right)\frac{1}{3} = \frac{3}{20} + \frac{1}{12} = \frac{9+5}{60} = \frac{14}{60} = \mathbf{\frac{7}{30}}$.

4. The sum of the contents of all 16 cells includes 8 copies of each of a, b, c, and d. For example, for the a's the cells with a are those in the a row and those in the a column (being careful not to double-count the cell where the a row and column meet), so $(a + a) + (a + b) + (a + c) + (a + d) + (b + a) + (c + a) + (d + a)$ contains 8 copies of a being added. Therefore, the sum of all the cells is $8a + 8b + 8c + 8d = 8a + 8c + 8(b + d)$. Now, $a + a = 2a = n + 10$, so $8a = 4n + 40$; $c + c = 2c = n + 6$ so $8c = 4n + 24$; $b + d = n + 3$ so $8(b + d) = 8n + 24$. Therefore the sum of all the cells is $8a + 8c + 8(b + d) = 4n + 40 + 4n + 24 + 8n + 24 = 16n + 88$, so $k = \mathbf{88}$.

5. The mean of the 5 values 5, 8, 10, 16 and n is $\frac{n+5+8+10+16}{5} = \frac{1}{5}n + \frac{39}{5}$.
The median is 8 if $n < 8$; 10 if $n > 10$; n if $8 \le n \le 10$. For the median to equal the mean, we, therefore, have three cases to consider:
1. $n < 8$: $8 = \frac{1}{5}n + \frac{39}{5}$ means that $8 - \frac{39}{5} = \frac{1}{5}n \longrightarrow \frac{1}{5} = \frac{1}{5}n$ so, $n = 1$.
2. $8 \le n \le 10$: $n = \frac{1}{5}n + \frac{39}{5}$ means that $\frac{4}{5}n = \frac{39}{5}$, so $n = \frac{39}{4}$.
3. $10 < n$: $10 = \frac{1}{5}n + \frac{39}{5}$ means that $10 - \frac{39}{5} = \frac{1}{5}n \longrightarrow \frac{11}{5} = \frac{1}{5}n$ so, $n = 11$.
All of these values satisfy the constraints on n. The sum of these three values is $1 + \frac{39}{4} + 11 = \frac{48+39}{4} = \mathbf{\frac{87}{4}}$.

6. The area enclosed by a trapezoid is the product of the height times the average of the base lengths. Therefore, $A = 12\left(\frac{7.4 + x + 15.6 - x}{2}\right) = 6 \times 23 = \mathbf{138}$ units^2.

7. ABFE is a square of side length 5, so encloses area of 5^2.
BCHG is a square of side length 12, so encloses area of 12^2.
ACID is a square of side length 13, so encloses area of 13^2.
ABC is a right triangle enclosing area of $\frac{1}{2}(5 \times 12) = 30$.
Triangle FBG is congruent to ΔABC, so encloses area of 30.
Let side EA be the base of ΔEAD, and it has length 5.
Triangle DXA, which is congruent to ΔABC, has been included to show that the perpendicular height of ΔEAD is 12, so the area enclosed by ΔEAD is also 30.
Let side HC be the base of ΔHCI, and it has length 12.
Triangle CYI, which is also congruent to ΔABC, has been included to show that the perpendicular height of ΔHCI is 5, so the area enclosed by HCI is also 30.
Therefore, the area enclosed by the hexagon is:
$5^2 + 12^2 + 13^2 + 4 \times 30 = 2 \times 13^2 + 120 =$ **458** units².



8. For $|x| \leq 1$, $y = x^2$; for $|x \geq 1|$, the reflection about $y = 1$ is given by $y = 2 - x^2$. Therefore, for a given value of y in the range $0 < y < 1$, we have four satisfying values of x: $-\sqrt{2-y}$, $-\sqrt{y}$, $\sqrt{y}$, and $\sqrt{2-y}$. For these to be equidistant, $\sqrt{2-y}$ has to be as much greater than $\sqrt{y}$ as $\sqrt{y}$ is greater than $-\sqrt{y}$. Thus, $\sqrt{2-y}$ must be $2\sqrt{y}$ greater than $\sqrt{y}$, so $\sqrt{2-y} = 3\sqrt{y}$. Squaring both sides yields $2 - y = 9y$, $10y = 2$, and $y = \frac{1}{5}$. The separation between two consecutive intersections is $2\sqrt{y} = 2\sqrt{1/5} = \mathbf{\frac{2\sqrt{5}}{5}}$ as a common fraction in simplest radical form.

2017 State Competition Team Round

1. The produce of the consecutive integers has the form $(n-1)(n)(n+1) = n^3 - n$, and the sum is $3n$. The value of n will be slightly larger than the cube root of the value of the product. Now, $\sqrt[3]{157{,}410}$ is approximately 53.994, so let's round up to n = 54 and check it out. Indeed $53 \times 54 \times 55 = 157{,}410$ and the desired sum is $3n = 3 \times 54 = \mathbf{162}$.

2. The starting amounts are $\frac{3}{3+2}(100 \text{ kg}) = 60$ kg of regular and $100 \text{ kg} - 60 \text{ kg} = 40$ kg of premium. We want to keep the same amount of premium and add regular until there is $4/1 = 4$ times as much regular as there is premium. Therefore, a total of $4 \times 40 \text{ kg} = 160$ kg of regular is needed, so $160 \text{ kg} - 60 \text{ kg} = \mathbf{100}$ kg of regular needs to be added.

3. None of the coordinates is permitted to have a magnitude of 2 or more. Neither is it permitted for all three coordinates to have magnitude 1. This leaves three permitted cases:
   1. 3 coordinates with value 0 and 0 with magnitude 1: There is only 1 option: (0, 0, 0).
   2. 2 coordinates with value 0 and 1 with magnitude 1: The basic form is $(0, 0, \pm 1)$, which is 2 options. However, the ± 1 can be placed in any of the 3 coordinate positions, which multiplies the number of options by 3 to make 6 options for this case.
   3. 1 coordinate with value 0 and 2 with magnitude 1: The basic form is $(0, \pm 1, \pm 1)$, which is 4 options. However, the 0 can be placed in any of the 3 coordinate positions, which multiplies the number of options by 3 to make 12 options for this case.

   The total number of options for points is $1 + 6 + 12 = \mathbf{19}$.

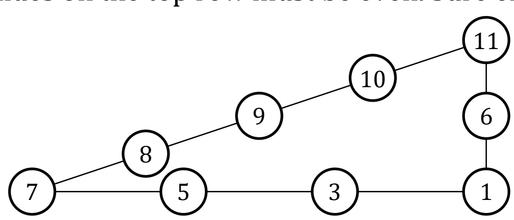
4. The slope of QR is $\frac{9-3}{4-1} = \frac{6}{3} = 2$. A line perpendicular to QR must have a slope that is the negative reciprocal of the slope of QR, thus $-\frac{1}{2}$. To go through P, the line must satisfy $\left(-\frac{1}{2}\right)12 + b = 10$, and b is the quantity we seek the value of. $-6 + b = 10$, so $b = \mathbf{16}$.

5. For convenience let's work with $a \geq b$ and $c \geq d$. Since 420 is divisible by 7, a needs to be at least 7. With $a = 7$, $a! = 5040$ and $5040/420 = 12$, so $\frac{c!d!}{b!} = 12$. Now, 12 can be expressed as $6 \times 2 = \frac{3!2!}{0!}$ (remembering that non-negative integers are specified, and $0! = 1$—not just 1! equals 1) and $\frac{24}{2} = \frac{4!0!}{2!}$. The former case has $b + c + d = 5$, which is less than the latter case with $b + c + d = 6$, so the least sum is $7 + 5 = \mathbf{12}$.

6. The 2017th dollar must be paid with a \$1 bill, so we can look at how to account for the other \$2016. Let f be the number of \$4 bills, t the number of \$2 bills, and o the number of \$1 bills. Then $4f + 2t + o = 2016$, where f, t, and o are non-negative integers. Therefore, $0 \leq f \leq 504$, the upper value being when there are no \$2 nor \$1 bills. When there are f \$4 bills, t must satisfy $0 \leq t \leq \frac{2016-4f}{2} = 1008 - 2f$, thus, $1009 - 2f$ choices, with the rest being \$1 bills. So, starting with $f = 504$ and decrementing by 1s to 0 yields the sequence of count of options for t: 1; 3; 5; …; 1007; 1009. This is the sequence of the first 505 positive odd integers, which has sum $505^2 = \mathbf{255{,}025}$.

7. Let $m\angle BAC = m\angle ABC = x$. Since segment AP bisects $\angle BAC$, we know that $m\angle BAP = m\angle CAP = \frac{1}{2}x$. Also, since $\Delta ABP \sim \Delta DBQ$, we have $m\angle ABC = m\angle ABP = m\angle QBD = x$ and $m\angle BAP = m\angle BDQ = \frac{1}{2}x$. Similarly, since segment DR bisects $\angle BDQ$, we have $m\angle BDR = m\angle QDR = \frac{1}{4}x$. By theorem, the sum of the angles of a triangle must be 180°. So, $m\angle BPA = m\angle BQD = 180° - \left(x + \frac{1}{2}x\right) = 180° - \frac{3}{2}x$ and $m\angle DRB = 180° - \left(x + \frac{1}{4}x\right) = 180° - \frac{5}{4}x$. Now, let E be the intersection of bisectors AP and DR. Again, by theorem, the sum of the angles of a quadrilateral must be 360°. We know that $m\angle BPE = m\angle BPA = 180° - \frac{3}{2}x$, $m\angle ABP = m\angle RBP = x$ and $m\angle PER = 90°$, and $m\angle DRB = m\angle ERB = 360° - \left(180° - \frac{3}{2}x + x + 90°\right) = 90° + \frac{1}{2}x$. We now have two equations for $m\angle DRB$, which must equal each other. Therefore, $180° - \frac{5}{4}x = 90° + \frac{1}{2}x$. Solving for x, which is the degree measure of $\angle ABC$, we get $90° = \frac{7}{4}x \longrightarrow x = 90° \times \frac{4}{7} = \mathbf{\frac{360}{7}}$ degrees.

8. Every other term in an arithmetic sequence must have the same parity (oddness or evenness) since term with value a is followed by $a + d$ and then $a + 2d$—when two terms differ by an even number, they must have the same parity. Therefore, the following terms must have the same parity: the lower right corner, the upper right corner, the middle on the top row, the left corner, and the other two in the bottom row because of the opposing lower corner. That means at least 6 distinct values with the same parity, i.e., at least 6 odd values (the largest having to be at least 11) or at least 6 even values (the largest having to be at least 12). Let's be optimistic and try the smaller case first with the 6 smallest positive odd integers 1, 3, 5, 7, 9, and 11 along with 3 positive even integers less than 11 (with 5 to choose from). The bottom row must be all odds, as well as the upper right corner and the middle of the top row; the middle of the right row and the other two values on the top row must be even. Sure enough, **11** works:



9. Let L be the length of the pool. At the first passing, Eddie has gone 72 ft while Missy has gone $L - 72$ ft. At the next passing Eddie has turned around at the end of his first lap (where Missy started) and gone 40 ft extra, while Missy will be the same 40 ft short of a full round trip; thus, Eddie has gone $L + 40$ ft and Missy has gone $2L - 40$ ft. Since each is moving at a constant rate, the ratio of the second distance to the first distance is the same for each: $\frac{L+40 \text{ ft}}{72 \text{ ft}} = \frac{2L-40 \text{ ft}}{L-72 \text{ ft}}$. Cross-multiplying yields $L^2 - 32L \text{ ft} - 2880 \text{ ft}^2 = 144L \text{ ft} - 2880 \text{ ft}^2$, reducing to $0 = L^2 - 176L \text{ ft} = L(L - 176 \text{ ft})$. Thus, either $L = 0$, which is absurd, or $L = \mathbf{176}$ ft, the latter being the only viable option.

10. Place the triangle on the xy-plane with BC on the x-axis and AB on the y-axis. Then point Q is at (4, 0) and P is at (8, 0). Since point A is at (0, 5), the lines AQ and AR are given by equations $y = 5 - \frac{5}{4}x$ and $y = 5 - \frac{5}{8}x$, respectively. Since BP bisects the right angle at the origin, the line BP is given by $y = x$. Point X is at the intersection of $y = x$ and $y = 5 - \frac{5}{4}x$, which occurs at $\left(\frac{20}{9}, \frac{20}{9}\right)$; similarly, point Y is at the intersection of $y = x$ and $y = 5 - \frac{5}{8}x$, which occurs at $\left(\frac{40}{13}, \frac{40}{13}\right)$. Thus, the coordinates of quadrilateral QRYX are:

Q(4, 0)	$4 \times 0 - 0 \times 8 = 0$
R(8, 0)	$8 \times \frac{40}{13} - 0 \times \frac{40}{13} = \frac{320}{13}$
$\text{Y}\left(\frac{40}{13}, \frac{40}{13}\right)$	$\frac{40}{13} \times \frac{40}{13} - \frac{40}{13} \times \frac{40}{13} = 0$
$\text{X}\left(\frac{20}{9}, \frac{20}{9}\right)$	$\frac{20}{9} \times 0 - \frac{20}{9} \times 4 = -\frac{80}{9}$

We can use the surveyor's algorithm, commonly called the shoelace method in MATHCOUNTS, to find the enclosing area. This algorithm requires repeating the first point in the list at the end, which is why Q appears twice above. The text boxes to the right of the points show the determinant calculation for each pair of consecutive points. The enclosed area is one-half the absolute value of the sum of the determinants, so $\frac{\left|0+\frac{320}{13}+0-\frac{80}{9}\right|}{2} = \mathbf{\frac{920}{117}}$.