MATHCOUNTS®

2017 Chapter Competition Solutions

Are you wondering how we could have possibly thought that a Mathlete® would be able to answer a particular Sprint Round problem without a calculator?

Are you wondering how we could have possibly thought that a Mathlete would be able to answer a particular Target Round problem in less 3 minutes?

Are you wondering how we could have possibly thought that a particular Team Round problem would be solved by a team of only four Mathletes?

The following pages provide solutions to the Sprint, Target and Team Rounds of the 2017 MATHCOUNTS® Chapter Competition. These solutions provide creative and concise ways of solving the problems from the competition.

There are certainly numerous other solutions that also lead to the correct answer, some even more creative and more concise!

We encourage you to find a variety of approaches to solving these fun and challenging MATHCOUNTS problems.

Special thanks to solutions author
Howard Ludwig
for graciously and voluntarily sharing his solutions with the MATHCOUNTS community.

2017 Chapter Competition Sprint Round

1. The real number 21.476 is not an integer. It is greater than 20.500 but less than 21.500, so **21** is the nearest integer.

2. The highest mark on the beaker is 500 mL, so the other numbers at the lower-level marks are also in terms of milliliters. The top of the blue region, corresponding to the amount of solution in the beaker, is very close to the 300 mark—certainly much closer than to the 200 and 400 marks. Therefore, to the nearest 100 mL, the volume of the solution in the beaker is **300** mL.

3. The first number Debbie says is 110;
the second number is 5 more, so 115;
the third number is 5 more, so 120;
the fourth number is 5 more, so 125;
the fifth number is 5 more, so **130**.
More generally, the nth number Debbie would say is $110 + 5(n - 1)$. Here $n = 5$.

4. Remember that multiplications are to be done before subtractions unless overridden by parentheses. We can explicitly write the parentheses to remind ourselves:
$7\times6 - 5\times4 = (7\times6) - (5\times4) = 42 - 20 = \mathbf{22}$.

5. One dozen eggs is 12 eggs. The eggs plus the carton weigh 32 oz, of which 2 oz is the carton, leaving 30 oz for the eggs. Since 12 eggs weigh 30 oz, the average weight for each egg is $\frac{30\text{ oz}}{12\text{ egg}} = \frac{6\times5}{6\times2}\frac{\text{oz}}{\text{egg}} = \frac{5}{2}\frac{\text{oz}}{\text{egg}} = \mathbf{2.5}$ ounces per egg.

6. $1\text{ cbl} = 240\text{ yd} = 240\,\cancel{\text{yd}}\times\frac{3\text{ ft}}{1\,\cancel{\text{yd}}} = \mathbf{720}\text{ ft}$.

7. Putting Velma's 7 scores in increasing order yields 84; 86; 88; 94; 95; 99; 100. For 7 scores, $(7 + 1)/2 = 4$ yields the fourth score being the median, with 3 below and 3 above. The fourth score is **94**.

8. Since 1/2 of the students joined, $1 - 1/2 = 1/2$ did not join, which constitutes 13 students. Therefore, the total number of students is $\frac{13}{1/2} = 13\times2 = \mathbf{26}$.

9. Remember that multiplications are to be done before subtractions unless overridden by parentheses. We can explicitly write the parentheses to remind ourselves:
$100 - \frac{10}{0.1} = 100 - \left(\frac{10}{0.1}\right) = 100 - \left(\frac{10\times10}{10\times0.1}\right) = 100 - \left(\frac{100}{1}\right) = 100 - 100 = \mathbf{0}$.

10. There are 5 gaps going from A through F corresponding to $(20 - 10)$ units $= 10$ units. Therefore, each gap is 10 units$/5 = 2$ units long. Going from C through F covers 3 gaps, so the length of that segment is $3\,\cancel{\text{gap}}\times\frac{2\text{ units}}{\cancel{\text{gap}}} = \mathbf{6}$ units.

11. Let n be the number of 6th graders. Then $n/4$ is the number of 8th graders. The two together total 100 students, so $100 = n + \frac{1}{4}n = \frac{5}{4}n$, so $n = \frac{4}{5}\times100 = \mathbf{80}$.

12. There are 12 "unit" triangles—six in the left half and six in the right half. The truss as a whole is 1 triangle. The only question left now is how many compound (bigger than unit) triangles but smaller than the whole truss. Starting from the left vertex of the truss, each segment connecting the bottom edge and the upper left edge of the truss combined with the bottom and upper left edges toward the left vertex form a triangle. The leftmost connecting segment forms a unit triangle so it must not be counted again. However, the other 5 connecting segments do form a triangle not yet counted; there are likewise 5 such compound triangles in the right half of the truss. Adding them together yields 12 + 1 + 5 + 5 = **23** triangles in total.

13. For each group of 5 cats there are 7 dogs, which is 2 more than the number of cats. Since we want 3 times that difference, we need 3 times as many cats, so 3 × 5 cats = **15** cats.

14. The base of the triangle is 5 in.
The area enclosed by the triangle is 30 in^2,
which is half the base times the height (h),
Therefore, 30 in^2 $= \frac{1}{2} \times 5 \text{ in} \times h$, so
$h = 2 \times \frac{30 \text{ in}^2}{5 \text{ in}} = 2 \times 6 \text{ in} = \mathbf{12}$ in.

15. $\left(\frac{1}{3}\right)^{-2} = \left(\frac{3}{1}\right)^{2} = 3^2 = \mathbf{9}$.

16. $a + 5 < 12$ and $a > 0$, so $0 < a < 12 - 5 = 7$. Thus, a can be any integer from 1 through 6, the sum of which would be the 6th triangular number, $6(6 + 1)/2 = 6 \times 7/2 = 3 \times 7 = \mathbf{21}$.

17. Since 90 students bike out of the total 360 students, this corresponds to a percentage of
$\frac{90}{360} \times 100\ \% = \frac{\cancel{90}}{\cancel{90}} \times \frac{1}{4} \times 100\ \% = \mathbf{25}\ \%$.

18. If the camel is carrying 25 small jugs, then the camel can carry 30 – 25 = 5 more small jugs or 5 small × 18 large/30 small = **3** large jugs.

19. $\frac{1}{2}\ \% = 0.5 \times 0.01 = \mathbf{0.005}$.

20. The quadratic $x^2 + 5x + 6 = 0$ factors to $(x + 2)(x + 3) = 0$. So, the solutions are $x = -2$ and $x = -3$, which sum to $-2 + (-3) = \mathbf{-5}$.

21. $\frac{1000}{2^3 \times 5^2} = \frac{1000}{2 \times 2^2 \times 5^2} = \frac{1000}{2 \times 10^2} = \frac{1000}{2 \times 100} = \frac{10}{2} = \mathbf{5}$.

22. The first $\frac{1}{2}$ mi costs \$2.25. For a 3 mi ride, there remains $2\frac{1}{2}$ mi $= \frac{5}{2}$ mi $= \frac{10}{4}$ mi, with each $\frac{1}{4}$ mi costing 75 ¢ = \$0.75. Thus, 10 times this distance is 10 times this price, or 10×\$0.75 = \$7.50. The total price is then \$2.25 + \$7.50 = \$**9.75**.

23. A total of 68 students are 7th graders, of which 40 are girls. The remaining $68 - 40 = 28$ are 7th-grade boys. The rest of the 54 total boys are 8th graders, thus $54 - 28 = \mathbf{26}$ boys in the 8th grade.

24. First aim for the largest number of thousands—use 7 from 2017 and 6 from 2016. Then aim for the largest number of hundreds using what is left, which must be a 2 and a 1, since we are not allowed to use the same digit twice for one place value. Then aim for the largest number of tens using what is left—that might seem like 1 and 2, but then we would have both 0's for the ones; the next best we can do is 0 and 2. That leaves 1 and 0 for the ones. Therefore: 7201 + 6120 = **13 321**.

25. $1 = \frac{x}{12} + \frac{y}{36} = \frac{3x+y}{36}$, which implies $3x + y = 36$. Since $3x$ and 36 are both divisible by 3, then so must be y as the difference of two multiples of 3. The only positive integers less than 36 (must keep $x > 0$) that are multiples of 3 are 1 × 3, 2 × 3, 3 × 3, ..., 11 × 3. Thus, there are **11** such values.

26. $$\begin{aligned} 55\times59 - 53\times57 &= (57-2)(57+2) - (55-2)(55+2) \\ &= 57^2 - 2^2 - (55^2 - 2^2) \\ &= 57^2 - 55^2 \\ &= (57+55)(57-55) \\ &= 112\times2 = 224 = 15^2 - 1, \end{aligned}$$
so $x = \mathbf{15}$.

27. A consequence of the two given patterns is $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = s^2$. Now, $s = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$Here we have $n = 8$, so $s = \frac{8\times9}{2} = \mathbf{36}$.

28. The radius r of a circle inscribed in a triangle is equal to the area A enclosed by the triangle divided by the semiperimeter s of the triangle. We are given that circle Q has enclosed are $4\pi\text{ in}^2 = \pi r^2$, so $r = 2$ in.
Let X, Y, and Z be the points of tangency of circle Q with segments AB, AC, and BC, respectively. BX = BZ $= r = 2$ in, and AY = AX = AB − BX = 6 in − 2 in = 4 in. Let d = CY = CZ. With the right triangle, we have $(\text{AB})^2 + (\text{BC})^2 = (\text{AC})^2$, so $(6\text{ in})^2 + (2\text{ in} + d)^2 = (4\text{ in} + d)^2$. Therefore, $36\text{ in}^2 + 4\text{ in}^2 + 4d\text{ in} + d^2 = 16\text{ in}^2 + 8d\text{ in} + d^2$, so $24\text{ in}^2 = 4d$ in, and $d = 6$ in. This means AB = 6 in, BC = 2 in + 6 in = 8 in, and AC = 4 in + 6 in = 10 in. Thus, we have a 3-4-5 right triangle with scale factor 2 in. The area enclosed by the triangle is $A = \frac{1}{2}\times6\text{ in}\times8\text{ in} = \mathbf{24}\text{ in}^2$.

29. The only ways to achieve 30 ¢ are:

3D	All 3D must be H and all 4N must be T—only 1 way to do this;
2D 2N	2DH, 1DT, 2NH, 2NT—C(3, 2) × C(4, 2) = 3× 6 = 18 ways to do this;
2D 1N 5P	2DH, 1DT, 1NH, 3NT—C(3, 2) × C(4, 1) = 3 × 4 = 12 ways to do this;
1D 4N	1DH, 2DT, 4NH—C(3, 1) × C(4, 4) = 3 × 1 = 3 ways to do this;
1D 3N 5P	1DH, 2DT, 3NH, 1NT—C(3, 1) × C(4, 3) = 3 × 4 = 12 ways to do this.

Total ways to achieve 30 ¢ is 1 + 18 + 12 + 3 + 12 =46; $2^{12} = 4096$ total options.
Therefore, 46/4096 = **23/2048**.

30. $a + b^2 + c^3 + d^4 + e^5 = 2$ requires either (1) two of the five terms to be 1 and the remainder be 0, or (2) three of the five terms have value 1, one term has value −1, and the remaining term has value 0.

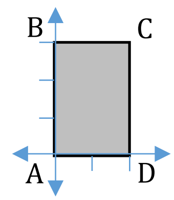
Case (1) has 3 subcases: (a) The two nonzero cases are chosen from a, c, and e. The two nonzero values must be 1 and the third value must be 0. There are 3 choices (a, c, and e) for which one is nonzero. (b) One of a, c, and e must be 1 and one of b and d must be +1 or −1, with the rest being 0, which constitutes 3 × 4 = 12 choices. (c) Each of b and d must be +1 or −1, with all other terms 0, which constitutes 4 choices. These 3 subcases add up to 19 possibilities for Case (1).

Case (2) has 2 subcases: (a) Each of b and d is +1 or −1 (contributing 4 options) and one of a, c, and e must be +1, one must be −1, and one must be 0 (contributing 6 options), so 24 total possibilities. (b) One of b and d is +1 or −1 and the other is 0 (contributing 4 options), with two of a, c, and e being +1 and the other being −1 (contributing 3 options), so 4 × 3 = 12 choices. These 2 subcases add up to 36 possibilities for Case (2).

Combining the 2 cases yields 19 + 36 = **55** total possibilities.

2017 Chapter Competition Target Round

1. The area enclosed by the rectangle is the product of the width times height.
The width is 2 units − 0 units = 2 units and height is 3 units − 0 units = 3units,
so the area is 2 units × 3 units = **6** units^2.



2.

$0.5^3 = 0.125$

$0.3^3 = 0.027$

$3^5/5^3$
$= 1.944$

0.5×0.3
$= 0.150$

$0.3 \div 0.5$
$= 0.600$

The greatest absolute difference occurs with the greatest value minus the least value, which is $1.944 - 0.027 =$ **1.917**.

3. Let a = # adult tickets sold and s = # student tickets sold. We are to find the value of a, given:
$a + s = 139$;
$\$13.50a + \$8.50s = \$1576.50$, which can be divided through by \$ to remove the \$ signs.
Therefore,

$$\begin{aligned} 1576.5 &= 13.5a + 8.5s \\ &= (5 + 8.5)a + 8.5s \\ &= 5a + 8.5a + 8.5s \\ &= 5a + 8.5(a + s) \\ &= 5a + 8.5(139) \\ &= 5a + 1181.5. \end{aligned}$$

Therefore, $5a = 1576.5 - 1181.5 = 395$, so $a = 395/5 =$ **79**.

4. There must be at least two 2's in order for 2 to be the mode. There cannot be more than two 2's, though, because that would make the median of four values be 2, which contradicts the given information. To have the median be greater than 2, the other two values must both be greater than 2. Therefore, in increasing order the four values must be 2; 2; a; b. It is the value of b that we are to find. For the median to be 4, we must have $(2 + a)/2 = 4$, so $2 + a = 8$ and $a = 6$. For the mean of all four values to be 5, we must have $(2 + 2 + 6 + b)/4 = 5$, so $10 + b = 20$ and $b =$ **10**.

5. Pay attention to units, especially for given information versus for final results. It helps to put all units into the calculation and manipulate them algebraically (just like variables) in order to keep them straight and compare with the units you ultimately need:

$$4000\ \cancel{\text{bill}} \times \frac{0.9\ \cancel{\text{oz}}}{25\ \cancel{\text{bill}}} \times \frac{1\ \text{lb}}{16\ \cancel{\text{oz}}} = 4000 \times \frac{0.9}{400}\ \text{lb} = 10 \times 0.9\ \text{lb} = \mathbf{9}\ \text{lb}.$$

6. $c = \frac{a+b}{2}$.

$$d = \frac{b+c}{2} = \frac{b + \frac{a+b}{2}}{2} = \frac{a+3b}{4}\ .$$

$$6 = \frac{c+d}{2} = \frac{\frac{a+b}{2} + \frac{a+3b}{4}}{2} = \frac{3a+5b}{8}\ .$$

Therefore, $48 = 3a + 5b$.

Clearly $(a; b) = (16; 0)$ satisfies this equation but $b > 0$ is required in the statement of the problem. Other integer solutions can be obtained by successively subtracting 5 from a and adding 3 to b [the coefficients of b and a, respectively, in $48 = 3a + 5b$]:
$(a; b) = (11; 3)$, which satisfies the last remaining criterion $a > b$. The next possibility is $(a; b) = (6; 6)$, which fails that criterion. Therefore, the only integer solution satisfying all the criteria is $(a; b) = (11; 3)$, so $a = \mathbf{11}$.

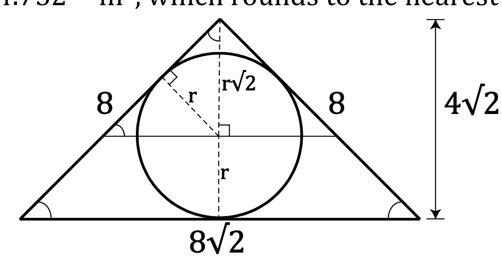
7. Jack must go down 3/5 of the hill while Jill goes up 2/5 of the hill. Jack must go a distance that is 1/5 of the hill greater than Jill. Since the hill is 1260 ft, Jack must travel (1260/5) ft = 252 ft more than Jill, and this occurs in 2 min = 120 s time (converting to seconds since the result is to be in feet per second). Therefore, Jack must be going faster than Jill by $\frac{252\text{ ft}}{120\text{ s}} = \mathbf{2.1}$ ft/s.

8. We want the probability that at least two of the ten students draw the same chip, which is the same as 1 minus the complementary probability, namely that no repeat selections are made. There are always 25 chips to draw from. For the second student, there are 24 chips to choose that do not match the selection of the previous student. For the third student, there are 23 chips to choose that do not match the selection of either of the previous two students. This process continues until the tenth student, for whom there are 16 chips to choose that do not match the selection of any of the previous nine students. Therefore, the probability of having no duplicate is $24\times23\times22\times21\times20\times19\times18\times17\times16/25^9 \approx 0.1244$, and the probability of having at least one duplicate is approximately $1 - 0.1244 = 0.8756$, which rounds to the nearest hundredth as **0.88**. (Note: If you have a factorial button on your calculator, calculating the numerator as $\frac{24!}{15!}$ will be a little easier and less prone to error.)

2017 Chapter Competition Team Round

1. The first statement requires the numbers to be in the sequence 1, 4, 7, 10, 13, 16, 19, 22, ...
The second statement requires the value to be greater than 1, so 1 is rejected and
4 is the first Lagado prime.
The next candidate is 7, which is not divisible by 4, so 7 is the second Lagado prime.
The next candidate is 10, which is not divisible by 4 or 7, so 10 is the third Lagado prime.
The next candidate is 13, which is not divisible by 4, 7, or 10, so 13 is the fourth Lagado prime.
The next candidate is 16, which is divisible by 4 (the first Lagado prime), so it is not a Lagado prime.
The next candidate is 19, which is not divisible by 4, 7, 10, or 13, so **19** is the fifth Lagado prime.

2. The radius of a circle inscribed in a triangle is the area enclosed by the triangle divided by the semiperimeter of the triangle. The area enclosed by the triangle is $\frac{1}{2}\times 8 \text{ in}\times 8 \text{ in} = 32 \text{ in}^2$. For the semiperimeter, we need to know the length of the hypotenuse, which is $8\sqrt{2}$ in. Therefore, the semiperimeter is $(8 \text{ in} + 8 \text{ in} + 8\sqrt{2} \text{ in})/2 = (8 + 4\sqrt{2})$ in, and the radius of the inscribed circle is $\frac{32 \text{ in}^2}{(8+4\sqrt{2}) \text{ in}} = 8 \text{ in}\times\frac{2-\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})} = 8 \text{ in}\times\frac{2-\sqrt{2}}{4-2} = 4 \text{ in}\times(2-\sqrt{2}) = 4(2-\sqrt{2})$ in. The area enclosed by a circle of such radius is $\pi\left[4(2-\sqrt{2}) \text{ in}\right]^2 = 16(6-4\sqrt{2})\pi \text{ in}^2 = 17.248\ldots \text{ in}^2$. The shaded area is, therefore, $32 \text{ in}^2 - 17.248\cdots \text{ in}^2 = 14.752\cdots \text{ in}^2$, which rounds to the nearest hundredth of a square inch as **14.75** in^2.



Note: the radius of the circle can also be found using similar triangles. From the figure, we can see that $r + r\sqrt{2} = 4\sqrt{2}$. Solving for r, we will find the radius is $r = 8 - 4\sqrt{2}$ inches.

3. To partition 15 into two positive integers with neither of them exceeding 10, we can have 10 + 5, 9 + 6, and 8 + 7. There are 4 card ranks (10, jack, queen, king) that have value 10 and 1 rank for each of the other values. Each rank has 4 choices based on suit. Therefore, there are 16 cards with value 10 and 4 cards for each of the values 5, 6, 7, 8, and 9. Thus, 10 + 5 has 16×4 options, 9 + 6 has 4×4 options, and 8 + 7 has 4×4 options, for a total of $(16 + 4 + 4)4 = 24 \times 4 =$ **96** total options.

4. Adding all positive integers from 1 through n constitutes an arithmetic series whose sum is given by $n(n+1)/2$, which equals $[(n + ½)^2 - ¼]/2$. [This is based on completion of a square.] If we multiply this by 2 we get ¼ less than a square (though not a "perfect" square), so let's multiply by another factor of 4 (so now a total factor of 8) to get a subtraction of 1: $4(n + ½)^2 - 1 = (2n+1)^2 - 1$, which is indeed 1 less than a perfect square. Therefore, the desired factor is **8**.

5. Since AP:PB = 2:3, P is $\frac{2}{2+3} = \frac{2}{5}$ of the way from A to B. To get from A to B requires moving $13 - (-7) = 20$ in the horizontal direction and $-11 - 4 = -15$ in the vertical direction; 2/5 of that movement is $20 \times 2/5 = 4 \times 2 = 8$ horizontally and $-15 \times 2/5 = -3 \times 2 = -6$ vertically. Therefore, P has horizontal coordinate $-7 + 8 = 1$ and vertical coordinate $4 + (-6) = -2$. The product of these two components is $1(-2) =$ **−2**.

6. Six consecutive whole numbers have the form $n, n+1, n+2, n+3, n+4$, and $n+5$, the sum of which is $6n+15=3(2n+5)$, where n is a non-negative integer, so 3 explicitly divides this value for all non-negative integers n. Now, how about $2n+5$? Well, $2n+5$ is 5 for $n=0$ and 7 for $n=1$, which have only 1 as a common factor. Therefore, the greatest positive integer guaranteed to divide $6n+15$ is **3**.

7. The least-valued sequence of 4 consecutive positive integers is 1, 2, 3, 4. The greatest-valued sequence with all values less than 200 is 196, 197, 198, 199. Therefore, there are 196 total sequences to consider as candidates. To have an integer n divisible by 2, $n+1$ divisible by 3, $n+2$ divisible by 4, and $n+3$ divisible by 5 requires having n leave a remainder of 0, 2, 2, and 2 when divided by 2, 3, 4, and 5, respectively. Clearly 2 satisfies the criteria for dividing by 3, 4, and 5; when 2 is divided by 2, it leaves a remainder of 0, so $n=2$ satisfies all four divisibility criteria. The next greater number that does so is the least common multiple of the divisors 2, 3, 4, and 5, which is $3 \times 4 \times 5 = 60$. Thus, the next values that work are 62, 122, and 182. (The next one, 242, is above the maximum allowed, 196.) Therefore, we have 4 desired values out of 196 candidates, so the probability is $4/196 =$ **1/49**.

8. In a 30°-60°-90° triangle the longer leg is $\frac{\sqrt{3}}{2}$ times as long as the hypotenuse. The hypotenuse constitutes a side of one square and the longer leg a side of a neighboring square. Therefore, each side of square B is $\frac{\sqrt{3}}{2}$ times as long as each side of square A. The area enclosed by similar polygons is proportional to the square of the length of corresponding sides, so the area enclosed by square B is ¾ the area enclosed by square A. Likewise, the area enclosed by square C is ¾ the area enclosed by square B, which is then $(¾)^2 = 9/16$ (still more than 1/2) the area of A. Likewise, the area enclosed by square D is ¾ the area enclosed by square C, which is then $(¾)^3 = 27/64$ the area enclosed by square A. This is the first one that has less than ½ the area enclosed by square A, so **D** is the answer.

9. For two greater digits to have a sum that is a prime number, the sum must be odd (1 + 1 = 2 is an even prime sum of two digits, but almost certainly we will have a larger value, in which case the sum must be odd). For the sum to be odd, one digit must be even and the other digit odd. For the number with such digits to be prime, the even digit must be in the 10s place and the odd digit in the 1s place. Let's start with the greatest possibility—the 80s. The largest prime number in the 80s is 89, and 8 + 9 = 17, which is prime, so **89** works.

10. For n^2+55 to be a perfect square, it must be equivalent to a form:
$(n+m)^2 = n^2+2mn+m^2$ with m a positive integer. Therefore, 55 must be of the form
$2n+1, 4n+4, 6n+9, 8n+16, 10n+25$, or $12n+36$ (any larger value yielding $n<1$).
Expressions with both coefficients even cannot yield the odd value 55, so let's consider those with an odd constant term:
$2n+1=55$ works for $n=27$;
$6n+9=55$ has n failing to be an integer;
$10n+25=55$ works for $n=3$.
The sum of the working values is $27+3=$ **30**.