May 2013 Activity Solutions

Warm-Up!

1. Let's begin with the four ways 9 can be written as a sum of two positive integers. Then we can determine all the ways the second addend can be written as a sum of two distinct positive integers that are different from the first addend. We have

1 + 8:	2 + 7:	3 + 6:	4 + 5:
1 + (2 + 6)	~~2 + (1 + 6)~~	~~3 + (1 + 5)~~	~~4 + (2 + 3)~~
1 + (3 + 5)	2 + (3 + 4)	~~3 + (2 + 4)~~	

Notice we can discard any group in which one or both of the last two addends is less than the first addend since we would have already counted that group. We see that 9 can be written as the sum of 1, 2 and 6; 1, 3, and 5; or 2, 3 and 4. For each of these groups of three integers the addends can be arranged in 3! = 6 different ways. Therefore, the total number of ways that 9 can be written as a sum of three distinct positive integers is 6 × 3 = **18** ways.

2. As with the previous problem, let's begin with the ways 12 can be written as a sum of two distinct positive integers, then, in each case, determine the ways the second addend can be written as a sum of two distinct positive integers that are different from the first addend. We have

1 + 11:	2 + 10:	3 + 9:
1 + (2 + 9)	2 + (3 + 7)	3 + (4 + 5)
1 + (3 + 8)	2 + (4 + 6)	
1 + (4 + 7)		
1 + (5 + 6)		

Again, notice we discarded the groups derived from 4 + 8 since any two distinct positive integers that are different from 4 and add to 8 will be less than 4. The same is true for groups derived from 5 + 7. We see that 12 can be written as the sum of seven different groups of three distinct positive integers. Each of these seven groups can be written in 3! = 6 different ways. Therefore, the total number of ways 12 can be written as a sum of three distinct positive integers is 6 × 7 = **42** ways.

3. Like writing ${}_6C_2$, the expression $\binom{6}{2}$ is a way to represent '6 choose 2'. So $\binom{6}{2}$ = 6!/(4!2!) = **15**.

4. Unlike problems 1 and 2, here we are interested in the ways 7 can be written as a sum of three positive integers that are not necessarily distinct. Let's start with the ways 7 can be written as a sum of two positive integers, then, in each case, determine the ways the second addend can be written as the sum of two positive integers. We have

1 + 6:	2 + 5:
1 + (1 + 5)	2 + (2 + 3)
1 + (2 + 4)	
1 + (3 + 3)	

We see that 7 can be written as the sum of four different groups of three positive integers. Each of the three sums 1 + 1 + 5, 1 + 3 + 3 and 2 + 2 + 3 can be written in 3 ways, while the sum 1 + 2 + 4 can be written in 3! = 6 ways. That's a total of 3 × 3 + 6 = 9 + 6 = **15** ways to write 7 as the sum of three positive integers.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

5. Let's begin with the ways 19 can be written as the sum of three distinct positive integers then, in each case, determine the ways the third addend can be written as a sum of two distinct positive integers that are different from the first two addends. We have

1 + 2 + 16:
- 1 + 2 + (3 + 13)
- 1 + 2 + (4 + 12)
- 1 + 2 + (5 + 11)
- 1 + 2 + (6 + 10)
- 1 + 2 + (7 + 9)

1 + 3 + 15:
- 1 + 3 + (4 + 11)
- 1 + 3 + (5 + 10)
- 1 + 3 + (6 + 9)
- 1 + 3 + (7 + 8)

1 + 4 + 14:
- 1 + 4 + (5 + 9)
- 1 + 4 + (6 + 8)

1 + 5 + 13:
- 1 + 5 + (6 + 7)

2 + 3 + 14:
- 2 + 3 + (4 + 10)
- 2 + 3 + (5 + 9)
- 2 + 3 + (6 + 8)

2 + 4 + 13:
- 2 + 4 + (5 + 8)
- 2 + 4 + (6 + 7)

3 + 4 + 12:
- 3 + 4 + (5 + 7)

We see that 19 can be written as the sum of 18 different groups of four distinct positive integers. Each of these 18 groups can be written in 4! = 24 different ways. Therefore, 19 can be written as the sum of four distinct positive integers in 24 × 18 = **432** ways.

6. Let's begin with the ways 24 can be written as the sum of three positive integers where the first and second addends are the same. Then we can determine the ways the third addend can be written as the sum of two distinct positive integers that are different from the first two addends. We have

1 + 1 + 22:	10 different pairs of distinct positive integers that add to 22, less the pair 1 + 21 leaves <u>9</u> pairs
2 + 2 + 20:	9 different pairs that add to 20, less the pair 2 + 18 leaves <u>8</u> pairs
3 + 3 + 18:	8 different pairs that add to 18, less the pair 3 + 15 leaves <u>7</u> pairs
4 + 4 + 16:	7 different pairs that add to 16, less the pair 4 + 12 leaves <u>6</u> pairs
5 + 5 + 14:	6 different pairs that add to 14, less the pair 5 + 9 leaves <u>5</u> pairs
6 + 6 + 12:	<u>5</u> different pairs that add to 12
7 + 7 + 10:	4 different pairs that add to 10, less the pair 3 + 7 leaves <u>3</u> pairs
8 + 8 + 8:	<u>3</u> different pairs that add to 8
9 + 9 + 6:	<u>2</u> different pairs that add to 6
10 + 10 + 4:	<u>1</u> pair that adds to 4.

In each of the 9 + 8 + 7 + 6 + 5 + 5 + 3 + 3 + 2 + 1 = 49 different groups of four positive integers, two of the four integers are the same, which means each group can be written in 4!/2! = 12 different ways. Therefore, 24 can be written as the sum of four positive integers such that exactly two of the integers are the same in 12 × 49 = **588** ways.

7. Using the method from the video, we can consider 19 objects being partitioned into 5 groups. There are 18 places to put each of 4 partitions. Therefore,19 can be written as the sum of five positive integers in ${}_{18}C_4$ = 18!/(14!4!) = **3060** ways.

8. We can use the method from the video, but we need to account for the fact that we are only including even numbers. Since 2 × 15 = 30, we can consider 15 objects (2s) that we want to partition into 5 groups. For example, 2 + 2 + 2 ┃2 + 2 + 2┃2 + 2┃2 + 2 + 2 + 2┃2 + 2 + 2 represents the sum 6 + 6 + 4 + 8 + 6 = 30. So there are 14 places to put each of 4 partitions (a "+" will be inserted in the remaining 10 spaces). Thus, 30 can be written as the sum of five positive even integers in ${}_{14}C_4$ = 14!/(10!4!) = **1001** ways.

Let's use an alternative method to see if this is the correct answer. We begin with the ways 30 can be written as the sum of four positive even integers. Then we can determine the ways the fourth addend can be written as the sum of two positive even integers. We have

(A) 2 + 2 + 2 + 24: 6 different pairs of positive even integers that add to 24
(B) 2 + 2 + 4 + 22: 4 different pairs that add to 22 that were not previously counted
(C) 2 + 2 + 6 + 20: 3 different pairs that add to 20 that were not previously counted
(D) 2 + 2 + 8 + 18: 1 pair that adds to 18 that was not previously counted
(E) 2 + 4 + 4 + 20: 4 different pairs that add to 20 that were not previously counted
(F) 2 + 4 + 6 + 18: 2 different pairs that add to 18 that were not previously counted
(G) 2 + 4 + 8 + 16: 1 pair that adds to 16 that was not previously counted
(H) 2 + 6 + 6 + 16: 2 different pairs that add to 16 that were not previously counted
(I) 4 + 4 + 4 + 18: 3 different pairs that add to 18 that were not previously counted
(J) 4 + 4 + 6 + 16: 2 different pairs that add to 16 that were not previously counted
(K) 4 + 6 + 6 + 14: 1 pair that adds to 14 that was not previously counted
(L) 6 + 6 + 6 + 12: 1 pair that adds to 12 that was not previously counted

There are 6 + 4 + 3 + 1 + 4 + 2 + 1 + 2 + 3 + 2 + 1 + 1 = 30 different groups of five positive even integers that add to 30. The group with five 6s can be written 5!/5! = 1 way, and the group with no repeated addends can be written 5! = 120 ways. The sum of any group with 1 addend occurring two, three or four times can be written 5!/2! = 60, 5!/3! = 20 or 5!/4! = 5 ways, respectively. The sum of any group containing two addends that each occur twice can be written 5!/(2!2!) = 30 ways. Finally, the sum of the group containing three 2s and two 12s can be written 5!/(3!2!) = 10 ways. Based on these scenarios, for each of the groups (A) through (L) above we have

(A) 6 different pairs can be written in a total of 5 + 20 + 20 + 20 + 20 + 10 = <u>95</u> ways
(B) 4 different pairs can be written in a total of 30 + 60 + 60 + 60 = <u>210</u> ways
(C) 3 different pairs can be written a total of 30 + 30 + 60 = <u>120</u> ways
(D) 1 pair can be written <u>30</u> ways
(E) 4 different pairs can be written in a total of 20 + 60 + 60 + 30 = <u>170</u> ways
(F) 2 different pairs can be written in a total of 60 + 120 = <u>180</u> ways
(G) 1 pair can be written <u>20</u> ways
(H) 2 different pairs can be written in a total of 20 + 30 = <u>50</u> ways
(I) 3 different pairs can be written in a total of 5 + 20 + 20 = <u>45</u> ways
(J) 2 different pairs can be written in a total of 30 + 30 = <u>60</u> ways
(K) 1 pair can be written <u>20</u> ways
(L) 1 pair can be written <u>1</u> way.

Therefore, 30 can be written as the sum of five positive even integers in 95 + 210 + 120 + 30 + 170 + 180 + 20 + 50 + 45 + 60 + 20 + 1 = **1001** ways.