MATHCOUNTS Minis

April 2013 Activity Solutions

Warm-Up!

1. Since segments MN and OP are parallel, we can conclude that $\Delta MNQ \sim \Delta POQ$ (Angle-Angle). Therefore, the ratios of corresponding sides of the triangles are congruent. Since ON = 24 units, it follows that OQ = 24 − NQ. We can set up the following proportion: NQ/(24 − NQ) = 12/20. Cross-multiplying and solving for NQ, we get 20(NQ) = 12(24 − NQ) → 20(NQ) = 288 − 12(NQ) → 32(NQ) = 288 → NQ = **9** units.

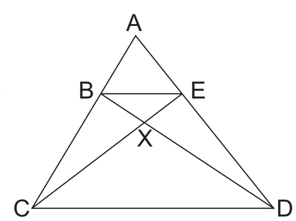
2. Since $\Delta SPQ \sim \Delta STU$ (Angle-Angle), the ratios of corresponding sides of the triangles are congruent. We are told that SP = 2PT → ½(SP) = PT. Since ST = SP + PT, we can write ST = SP + ½(SP) → ST = 3/2(SP) → SP/ST = 3/2. Since the ratio of corresponding sides of the triangles is 2/3, the ratio of the area of ΔSPQ to the area of ΔSTU is $2^2/3^2 = 4/9$. We also are told that the area of ΔSTU is 45 cm^2. So, it follows that the area of ΔSPQ = 4/9(45) = **20** cm^2.

3. The formula for the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where h and r are the height and the radius of the base of the cone, respectively. We are told that the circumference ($2\pi r$) of the base of the cone is 6π inches, thus r = 3 inches. Since the height of the cone is three times is radius, h = 3(3) = 9 inches. We now can substitute to see that the volume of the cone is $\frac{1}{3}\pi(3^2)(9)$ = $\mathbf{27\pi}$ in^3.

The Problem is solved in the MATHCOUNTS Mini.

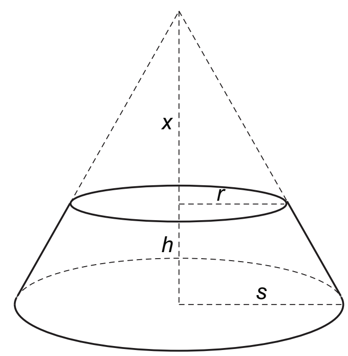
Follow-up Problems

4. Since QR = QU + UR and we are told that QR = 4, we have 4 = QU + UR → UR = 4 − QU. For similar triangles PQR and TUR, we can write the following proportion: 4/(4 − QU) = 3/UT. Because QSTU is a square, it follows that SQ = QU = UT = TS. Substituting, we get 4/(4 − QU) = 3/QU. Cross-multiplying and solving, we see that 4(QU) = 3(4 − QU) → 4(QU) = 12 − 3(QU) → 7(QU) = 12 → QU = **12/7** units.



5. From the figure, we can see that the area of ΔACD is the sum of the areas of ΔABE and trapezoid BCDE. Also, we are told that the area of trapezoid BCDE is 8 times the area of ΔABE. It follows that the area of ΔACD is 9 times the area of ΔABE. That means the ratio of sides BE and CD is $\sqrt{1}/\sqrt{9} = 1/3$. Since segments BE and CD are also sides of triangles EBX and CDX, respectively, it follows that the ratio of the areas of ΔEBX and ΔCDX is $1^2/3^2 = 1/9$. The problem states that the area of ΔCDX is 27 $units^2$, so the area of ΔEBX is (1/9) × 27 = 3 $units^2$. Using the method from the video, we can determine the areas of ΔBCX and ΔDEX by multiplying $\sqrt{3} \times \sqrt{27} = \sqrt{81} = 9$. Therefore, ΔBCX and ΔDEX each have an area of 9 $units^2$. We now can calculate the area of trapezoid BCDE to be 3 + 27 + 9 + 9 = 48 $units^2$. Let's see if Harvey's trick results in the same answer. We have $(\sqrt{3} + \sqrt{27})^2 = (\sqrt{3} + 3\sqrt{3})^2 = (4\sqrt{3})^2 = 48$ $units^2$. So the area of ΔABE is (1/8) × 48 = 6 $units^2$. Thus, the area of ΔACD is 48 + 6 = **54** $units^2$. This also confirms our assertion that the area of ΔACD is 9 times the area of ΔABE since 9 × 6 = **54** $units^2$.

6. Because triangles PAB and PAD have the same height, it follows that the ratio of their areas is just the ratio of the lengths of their bases. We are told that the areas of triangles PAB and PCD are a^2 units^2 and b^2 units^2, respectively. So we see that the ratio of the sides BP/PD = $\sqrt{a^2}/\sqrt{b^2} = a/b$. If we let x represent the area of ΔPAD, we can set up the proportion $a/b = a^2/x$. Solving for x, we see that the area of ΔPAD is $ax = a^2b \rightarrow x = ab$ units^2. That means the area of ΔPBC is also ab units^2. We now have the following expression for the area of trapezoid ABCD: $a^2 + ab + ab + b^2$. Simplifying, we get $a^2 + 2ab + b^2$, which factors to $\mathbf{(a + b)^2}$.



7. The figure shows a frustum with height h and bases of radius r and s. The frustum is created when the top of the cone, a smaller cone with height x and base of radius r, is removed from a larger cone with height $h + x$ and base of radius s. To determine the height of the smaller cone, we set up the proportion $(h + x)/s = x/r$. Cross-multiplying and solving for x, we get $hr + rx = sx \rightarrow hr = sx - rx \rightarrow hr = x(s - r) \rightarrow x = hr/(s - r)$. Using the formula for the volume of a cone, we see the volume of the larger cone is $\frac{1}{3}\pi s^2(h + x)$. Substituting the expression above for x, we get $\frac{1}{3}\pi s^2[h + (hr/(s - r))]$. We can simplify this expression to get $\frac{1}{3}\pi s^2[(h(s - r) + hr)/(s - r)] \rightarrow \frac{1}{3}\pi s^2[(hs - hr + hr)/(s - r)] \rightarrow \frac{1}{3}\pi s^2[(hs/(s - r)] \rightarrow (\pi s^3h)/[3(s - r)]$. The two similar triangles shown in the figure have sides in the ratio r/s. It follows that the ratio of the volume of the smaller cone to the volume of the larger cone is r^3/s^3. The volume of the smaller cone is r^3/s^3 of the volume of the larger cone. That means the volume of the frustum is $1 - r^3/s^3 = (s^3 - r^3)/s^3$ of the volume of the larger cone. Using our expression for the volume of the larger cone, we have the following for the volume of the frustum: $[(s^3 - r^3)/s^3] \times (\pi s^3h)/[3(s - r)]$. If we factor $s^3 - r^3$, which is the difference of cubes, we can rewrite the expession for the volume of the frustum and simplify to get $[(s - r)(s^2 + rs + r^2)/s^3] \times (\pi s^3h)/[3(s - r)] \rightarrow \mathbf{(s^2 + rs + r^2)(\pi h/3)}$.