February 2013 Activity Solutions

Warm-Up!

1. (a) 40 and 62 are **not relatively prime** since both are even numbers and, thus, each is divisible by 2.
   (b) 123 and 321 are **not relatively prime** since 1 + 2 + 3 = 3 + 2 + 1 = 6, which means both are divisible by 3.
   (c) $3 \cdot 5 \cdot 7 \cdot 9$ and $11 \cdot 13 \cdot 15 \cdot 17$ are **not relatively prime** since both are divisible by $3 \times 5 = 15$.
   (d) 1,234,567,890 and 1,234,567,891 are **relatively prime**.

2. The set contains 10 elements, so the total number of two-element subsets is ${}_{10}C_2 = 10!/(8!\ 2!) = (10 \times 9)/(2 \times 1) = 90/2 =$ **45**.

3. There exist **3** integers, x, less than 20 such that $2x$ is a perfect square: $2 \times 2 = 4 = 2^2$; $2 \times 8 = 16 = 4^2$ and $2 \times 18 = 36 = 6^2$.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

4. As in the video, we are looking for expressions of the form $2^2x^2 = 2(2x^2)$. That means we are looking for perfect squares, x^2, such that $2x^2 < 1000$. There are 22 perfect squares that meet this criteria: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441 and 484. When each of these perfect squares is doubled, we get the **22** positive integers less than 1000 such that 2 times the integer is a perfect square.

5. We can use the strategy from the video and try to construct a relatively prime pair of products. As in the video, one of the numbers must be odd. The product $1 \times 3 \times 5$ is the only product of three numbers in the set that is odd. Now we need to find the products of three numbers from the set that are not divisible by 3 and/or 5. Therefore, we can exclude 3, 5 and 6 (since $6 = 2 \times 3$). That leaves only one product $1 \times 2 \times 4$ to give us **1** pair of relatively prime products.

6. This is very similar to the previous problem, but now we have another odd number to consider, 7. There are 4 products of three different numbers from the set that are odd:

$1 \times 3 \times 5 = 15$ $\quad$ $1 \times 3 \times 7 = 21$ $\quad$ $1 \times 5 \times 7 = 35$ $\quad$ $3 \times 5 \times 7 = 105$

For the product $1 \times 3 \times 5 = 15$, we need to consider products that are not divisible by 3 and/or 5. Thus, we cannot use 3, 5 or 6. There are 4 products that are relatively prime to 15. They are

$1 \times 2 \times 4 = 8$ $\quad$ $1 \times 2 \times 7 = 14$ $\quad$ $1 \times 4 \times 7 = 28$ $\quad$ $2 \times 4 \times 7 = 56$

For the product 1 × 3 × 7 = 21, we need to consider products that are not divisible by 3 and/or 7. Thus, we cannot use 3, 7 or 6. There are 4 products that are relatively prime to 21. They are

1 × 2 × 4 = 8	1 × 2 × 5 = 10	1 × 4 × 5 = 20	2 × 4 × 5 = 40

For the product 1 × 5 × 7 = 35, we need to consider products that are not divisible by 5 and/or 7. Since 1, 2, 3, 4 and 6 are not divisible by 5 or 7, the product of any three of these numbers will be relatively prime to 35. That means there are ${}_5C_3$ = 5!/(3! 2!) = 10 such products. But we need to check to make sure each product creates a different pair of relatively prime numbers. The 10 products are

1 × 2 × 3 = 6	1 × 2 × 4 = 8	1 × 2 × 6 = 12	1 × 3 × 4 = 12
1 × 3 × 6 = 18	1 × 4 × 6 = 24	2 × 3 × 4 = 24	2 × 3 × 6 = 36
2 × 4 × 6 = 48	3 × 4 × 6 = 72		

Notice that there are two instances of duplicates: 1 × 2 × 6 = 1 × 3 × 4 = 12, and 1 × 4 × 6 = 2 × 3 × 4 = 24. Therefore, there are 10 − 2 = 8 products that are relatively prime to 35.

For the product 3 × 5 × 7 = 105, we must consider products that are not divisible by 3, 5 and/or 7. Thus, we cannot use 3, 5, 7 or 6. The product 1 × 2 × 4 = 8 is the only product that is relatively prime to 105.

Therefore, there are 4 + 4 + 8 + 1 = **17** different relatively prime pairs.

7. Let's use the Fundamental Counting Principle to count the number of four-digit numbers. Since the first digit can not be zero, there are 9 choices for the first digit. Since the second digit must be different from the first (and can be zero), there are also 9 choices for the second digit. Similiarly, there are 9 choices for the third digit and 9 choices for the fourth digit. That's a total of 9 × 9 × 9 × 9 = **6561** four-digit positive integers.

8. We are looking for quotients that are equal to perfect squares. Let's consider each perfect square and count the number of ways each can be expressed in the form *a*/*b* where *a* and *b* are distinct and both are less than 100. The only way to express 1 in this form is 1/1, in which case *a* = 1 and *b* = 1 and are not distinct.

So, we begin with 4, and 4 = 4/1 = 8/2 = 12/3 = ⋯ = 96/24, for a total 24 quotients.

For 9, we have 9 = 9/1 = 18/2 = 27/3⋯ = 99/11, for a total of 11 quotients.

For 16, we have 16 = 16/1 = 32/2 = 48/3 = ⋯ = 96/6, for a total of 6 quotients.

For 25, we have 25 = 25/1 = 50/2 = 75/3, for a total of 3 quotients.

For 36, we have 36 = 36/1 = 72/2, for a total of 2 quotients.

For 49, we have 49 = 49/1 = 98/2, for a total of 2 quotients.

Finally, for 64 and 81, we have 64 = 64/1 and 81 = 81/1.

Therefore, there are 24 + 11 + 6 + 3 + 2 + 2 + 1 + 1 = **50** such pairs of distinct integers.