MATHCOUNTS® Minis **January 2013 Activity Solutions**

Warm-Up!

1. Let n be the sum of the seven numbers. We are told that the average (mean) of the seven numbers is 43. It follows that $n \div 7 = 43 \rightarrow n = \mathbf{301}$.

2. We know that the largest of the seven numbers from the previous problem is 55, so let m be the sum of the remaining six numbers. We determine the sum of the remaining six numbers to be $m + 55 = 301 \rightarrow m = 246$. Therefore, the average of these six numbers is $246 \div 6 = \mathbf{41}$.

3. If we subtract the second equation from the first equation we get

$$\begin{array}{rl} u + v + w + x + y + z & = 45 \\ \underline{-\ (\ v + w + x + y + z} & \underline{= 21\ )} \\ u & = \mathbf{24} \end{array}$$

4. When we expand the given product we get $(a + b)(a + b) = a^2 + ab + ab + b^2 = \mathbf{a^2 + 2ab + b^2}$.

5. For the first number, let x and y be the numerator and denominator, respectively. For the second number, let u and v be the numerator and denominator, respectively. We are told that $\frac{x}{y} + \frac{u}{v} = 6$, which can be rewritten as $\frac{xv + uy}{yv} = 6 \rightarrow xv + uy = 6yv$. We are also told that $\frac{x}{y} \times \frac{u}{v} = 7$, which can be rewritten as $\frac{xu}{yv} = 7 \rightarrow xu = 7yv$. We are asked to find the sum of reciprocals of the two numbers, $\frac{y}{x} + \frac{v}{u}$, which can be rewritten as $\frac{uy + xv}{xu}$. Substituting, we have $\frac{uy + xv}{xu} = \frac{6yv}{7yv} = \mathbf{\frac{6}{7}}$.

A simpler way may be to let the two numbers be a and b. We are told that $a + b = 6$ and $ab = 7$. We are asked to find $\frac{1}{a} + \frac{1}{b}$. Getting a common denominator of ab, this can be rewritten as $\frac{a + b}{ab}$. Substituting our two unknown values, we see that $\frac{a + b}{ab} = \mathbf{\frac{6}{7}}$.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

6. Let's start by squaring each side of the given equation to get

$$\left(a+\frac{1}{a}\right)^2 = 3^2$$

$$a^2+1+1+\frac{1}{a^2} = 9$$

$$a^2+2+\frac{1}{a^2} = 9$$

$$a^2+\frac{1}{a^2} = \mathbf{7}.$$

7. Let's start by cubing each side of the given equation to get

$$\left(a+\frac{1}{a}\right)^3 = 3^3$$

$$\left(a+\frac{1}{a}\right)^2\left(a+\frac{1}{a}\right) = 27$$

$$\left(a^2+2+\frac{1}{a^2}\right)\left(a+\frac{1}{a}\right) = 27$$

$$a^3+a+2a+\frac{2}{a}+\frac{1}{a}+\frac{1}{a^3} = 27$$

$$a^3+3a+\frac{3}{a}+\frac{1}{a^3} = 27.$$

If we factor a 3 out of the two middle terms of the expression on the left-hand side of the equation, we have

$$a^3+3\left(a+\frac{1}{a}\right)+\frac{1}{a^3} = 27.$$

Since we know that $a + \frac{1}{a} = 3$, we can substitute and simplify to get

$$a^3+3(3)+\frac{1}{a^3} = 27$$

$$a^3+9+\frac{1}{a^3} = 27$$

$$a^3+\frac{1}{a^3} = \mathbf{18}.$$

8. Since we ultimately want the product of the three integers, let's start by multiplying the products of the pairs we are given. We have $(xy)(yz)(xz) = 4 \times 18 \times 50 \rightarrow x^2y^2z^2 = 3600 \rightarrow (xyz)^2 = 3600$. Now if we take the square root of each side, we get $xyz =$ **60**, since x, y and z are positive numbers.

9. Let n be the sum, and m be the mean of the seven numbers. We can write $n \div 7 = m \rightarrow n = 7m$. We are told that when one number, call it x, is removed from the list, the mean of the remaining six numbers is 5 less than the mean of the original seven numbers. That means $(n - x) \div 6 = m - 5 \rightarrow n - x = 6m - 30 \rightarrow n = 6m - 30 + x$. Since we have two expressions that are each equal to n, we can set them equal to one another. Doing so yields $7m = 6m - 30 + x$. With further algebraic manipulation, we get $m - x = -30$. That means the positive difference between the two quantities is $x - m =$ **30**.

10. In fact, it's not a coincidence. Consider the following list of seven numbers: 2, 3, 5, 7, 11, 14, 21. The mean of these seven numbers is 9. Let's find the difference between each of the seven numbers and the mean. The seven differences are −7, −6, −4, −2, 2, 5, 12. Notice that the sum of these differences is $-7 + (-6) + (-4) + (-2) + 2 + 5 + 12 = 0$. The sum of the differences in the mean of list of numbers and each number in the list must always be 0. So, if we remove 21 from our original list, the mean of the remaining six numbers is 7, which is 2 less than our original mean. The difference between the original mean and the number that was removed is 12. Notice that the sum of the differences of the each of the remaining six numbers and the original mean is $7 + (-6) + (-4) + (-2) + 2 + 5 = -12$. It makes sense that the difference between the number that was removed and the original mean must be $6 \times 2 = 12$ because the sum of the differences in the original mean and each number in the orignal list must be 0.