MATHCOUNTS Minis December 2012 Activity Solutions

Warm-Up!

1. There are 5 different color choices for the wall paint. Since a different color is to be chosen for the trim, then there are 4 color choices for the trim. That means there are 5 × 4 = **20** ways to choose one color for the walls and a different color for the trim.

2. In this case we need to determine the number of distinct pairs of marbles selected from the five marbles. When selecting the first marble, there are 5 possible outcomes, and there are 4 choices for the second marble. That's a total of 5 × 4 = 20 outcomes. But each of these outcomes is counted twice, since selecting color A then color B is the same as selecting color B then color A. Therefore, there are 20 ÷ 2 = **10** ways that two marbles can be chosen from the five to give as a present.

3. In the first problem, we selected 2 out of 5 possible colors. But the order in which we select color A and color B yielded two different outcomes. The first is if the wall was painted color A and the trim was painted color B. The second is if the wall was painted color B and the trim was painted color A. Here the order in which the colors are selected mattered, and in this case, we were counting *permutations* of 5 objects taken 2 at a time. Using the formula ${}_nP_r = n!/(n - r)!$, we have ${}_5P_2 = 5!/3! = (5 \times 4 \times 3 \times 2 \times 1)/(3 \times 2 \times 1) = 5 \times 4 = 20$.

In the second problem, we selected 2 out of 5 different colored marbles. Selecting a marble of color A followed by a marble of color B yielded the same outcome as selecting a marble of color B followed by a marble of color A. Since each of these selections resulted in the same outcome, order was not important. In this case, we were counting *combinations* of 5 objects taken 2 at a time. Using the formula ${}_nC_r = n!/[(n - r)!r!]$, we have ${}_5C_2 = 5!/(3! \times 2!) = (5 \times 4 \times 3 \times 2 \times 1)/[(3 \times 2 \times 1)(2 \times 1)] = (5 \times 4)/(2 \times 1) = 20/2 = 10$.

4. There are 6 different children who can receive the first gift, followed by 5 different children who can receive the second gift, and so on. Therefore, the number of ways that six different gifts can be given to six different children is 6 × 5 × 4 × 3 × 2 × 1 = **720** ways.

5. There are three dice, and six outcomes when rolling each die. If we are interested in rolling three different numbers then there are 6 × 5 × 4 = 120 ways this can occur. The total number of outcomes when rolling three dice is 6 × 6 × 6 = 216 outcomes. Therefore, the probability of rolling three different numbers is 120/216 = **5/9**.

We can also consider the probability of two independent events that occur once the first die is rolled. When the second die is rolled, the probability of getting a different number than the first roll is 5/6. And when the third die is rolled the probability of getting a different number than the first two rolls is 4/6 = 2/3. Therefore, the probability of getting three different numbers is (5/6) × (2/3) = 10/18 = **5/9**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

6. **Method 1:** Let's first count the number of such four-digit numbers with a thousands digit of 1. Thus, one of the digits is 1, and the other digit, *k*, could be any of the other 9 digits. If the 1 is used just once, the only possible arrangement is 1*kkk*, and there are 9 such numbers. If the 1 is used three times (and still is in the thousands place), the only possible arrangements are 1*k*11, 11*k*1 and 111*k*. Thus, if the thousands digit is 1, we can create 4 numbers for each of the 9 values of *k*, which means there are 4 × 9 = 36 such numbers. However, our thousands digit is not limited to 1. This same counting process would work for each of the digits 1 through 9 when placed in the thousands place, so there are a total of 9 × 36 = **324** four-digit numbers with exactly three identical digits.

Method 2: One thing that makes this problem difficult is ensuring we don't accidentally put a zero in the thousands place of our four-digit number. Let's break the problem into two cases. ***Case 1*** is when there are no zeros used at all. We need pairs of digits to use that do not include a zero. There are "9 choose 2" = 9!/[(2!)(9 − 2)!] = 9!/(2!7!) = (9 × 8)/(2 × 1) = 36 pairs of numbers to consider. For each pair A,B, we can create 8 numbers: ABBB, BABB, BBAB, BBBA, BAAA, ABAA, AABA and AAAB. So there are a total of 36 × 8 = 288 numbers we can create when no digit is a zero. ***Case 2*** is when a zero is used. If one digit is 0, the other digit could be any of the other 9 digits. For each pair 0,A, we can create 4 numbers: A0AA, AA0A, AAA0, A000. So there are a total of 9 × 4 = 36 numbers we can create when there is a zero. This is a total of 288 + 36 = **324** four-digit numbers with exactly three identical digits.

7. **Method 1:** Consider a five-digit number of the form __ __ __ __ 0. There are 9 possible choices for each of the first, second, third and fourth digits. That's 9 × 9 × 9 × 9 = 6561 different five-digit numbers. But the zero can be in one of 4 different locations (second through fifth digits). Therefore, there are a total of 6561 × 4 = **26,244** such five-digit numbers.

Method 2: First let's count the number of four-digit numbers we can make with no zeros. Each of the four digits can be any digit 1-9, so there are 9 × 9 × 9 × 9 = 6561 such four-digit numbers. For each of these numbers, we can create a unique five-digit number with exactly one zero by inserting the zero into one of the four locations identified with an arrow below. (Notice a zero could not be placed at the very front of the four-digit number.) For each of the 6561 four-digit numbers, 4 different five-digit numbers can be created. This is a total of 6561 × 4 = **26,244** such five-digit numbers.

___ ↑ ___ ↑ ___ ↑ ___ ↑

8. **Method 1:** We need to arrange 7 different girls, 1 empty seat and 1 pair of boys. This is 9 different "objects." (Since the pair of boys cannot be separated, we consider them one object for now.) Nine distinguishable objects can be arranged in 9! = 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 362,880 ways. Remember, however, that for each of these arrangements, the pair of boys can be seated in two different orders (A,B or B,A), so there are 362,880 × 2 = 725,760 ways to seat the 9 kids in the 10 seats.

Method 2: First, consider the 7 girls and the 1 empty seat. There are 8! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 = 40,320 ways to arrange them. One such arrangement is shown below. There are 9 different placements for the boys (each indicated with an arrow), and there are two ways to arrange the boys in each of these placements. That's a total of 40,320 × 9 × 2 = **725,760** arrangements of the 9 students in 10 seats.

$$\begin{array}{ccccccccccccccccc} & G_2 & & G_4 & & G_6 & & G_5 & & E & & G_1 & & G_7 & & G_3 & \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \end{array}$$

9. **Method 1:** Let's think of the possible choices for each of the digits in a 7-digit palindrome. Since the first digit can not be zero, there are 9 possible choices for the first digit. The second, third and fourth digits can each be any number since we were not told the digits have to be distinct. So, there are 10 possible choices each for the second, third and fourth digits. The fifth digit must be the same as the third digit; the sixth must be the same as the second; and the seventh must be the same as the first. Thus, there is only one choice for each of the fifth, sixth and seventh digits. Therefore, there are are 9 × 10 × 10 × 10 × 1 × 1 × 1 = **9000** possible 7-digit palindromes.

Method 2: For every 4-digit number (ABCD), we can create a unique 7-digit palindrome (A,BCD,CBA), and the first 4 digits of every 7-digit palindrome create a unique 4-digit number. There are 9000 4-digit numbers from 1000 through 9999, and each of these can be made into one of the **9000** 7-digit palindromes.