# MATHCOUNTS Minis — October 2012 Activity Solutions

**Warm-Up!**

1. The first term of the sequence is 2, and each term that follows is 3 times the previous term. To get the second term, we multiply the first term by 3 once. To get the third term, we multiply the first term by 3 twice. It follows that to get the seventh term, we need to multiply the first term by 3 six times. In other words, we need to multiply the first term by $3^6$. Therefore, the seventh term of the geometric sequence is $2 \times 3^6 = 2 \times 729 =$ **1458**.

2. Figure 1 has 1 dot. Figure 2 has 3 dots, which is 2 more that the previous figure. Figure 3 has 6 dots, which is 3 more than the previous figure. Finally, Figure 4 has 10 dots, which is 4 more than the previous figure. Notice the pattern shown in the table below.

| FIGURE | DOTS |
|---|---|
| 1 | 1 |
| 2 | 1 + 2 = 3 |
| 3 | 1 + 2 + 3 = 6 |
| 4 | 1 + 2 + 3 + 4 = 10 |
| ⋮ | ⋮ |
| $n$ | $1 + 2 + 3 + 4 + \cdots + n$ |

Therefore, Figure 10 has 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = **55** dots. The numbers of dots in each figure form a sequence of numbers commonly referred to as the Triangular Numbers. There is a formula to determine the sum of the first $n$ positive integers. It is $1 + 2 + 3 + 4 + \cdots + n = n(n + 1)/2$. So, in this case, the number of dots in Figure 10, which represents the tenth Triangular Number, is 10(11)/2 = 110/2 = **55** dots.

3. Since S(19) is the sum of the first 19 positive integers, it follows that S(20) = S(19) + 20. Therefore, S(20) − S(19) = **20**.

**The Problem** is solved in the MATHCOUNTS Mini.

**Follow-up Problems**

4. In this sequence, the first term is 3, and each subsequent term is −2 times the previous term. We are given the first four terms, and we need to find the next five terms in order to determine the sum of the first nine terms. The next five terms are: $3 \times (-2)^4 = \underline{48}$, $3 \times (-2)^5 = \underline{-96}$, $3 \times (-2)^6 = \underline{192}$, $3 \times (-2)^7 = \underline{-384}$, $3 \times (-2)^8 = \underline{768}$. Therefore, the sum of the first nine terms of this sequence is 3 − 6 + 12 − 24 + 48 − 96 + 192 − 384 + 768 = **513**.

5. The first term in Jenny's list is $1^2 = 1$. The product of the first two terms in her list must be $2^2 = 4$. Therefore, the second term is 4 ÷ 1 = 4. The product of the first three terms must be $3^2 = 9$. So, the third term is 9 ÷ (1 × 4) = 9/4. The product of the first four terms must be $4^2 = 16$. It follows, then, that the fourth term is 16 ÷ (1 × 4 × (9/4)) = 16/9. Notice the pattern? The $n$th term in Jenny's list equals $n^2/(n - 1)^2$. So, the last term in Jenny's list, which is the 12th term, equals $12^2/11^2 =$ **144/121**.

6. When the dots are connected, triangles are created, as shown. Instead of looking at the line segments, let's look at the number of shaded triangles in each figure.

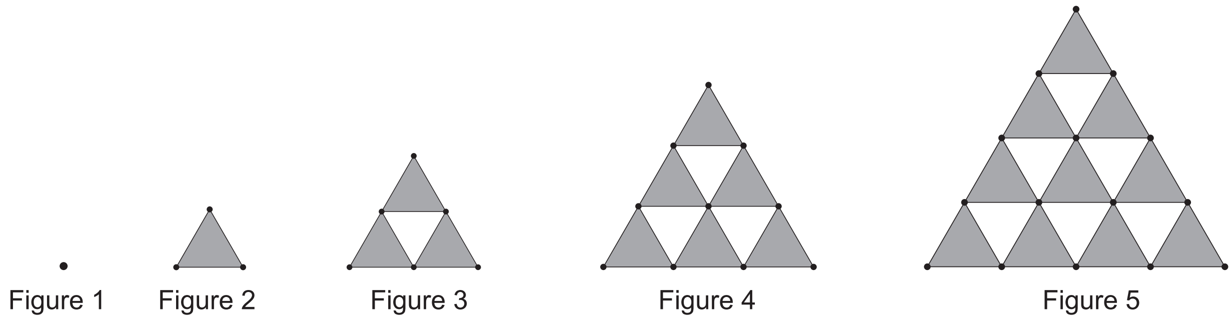


Figure 1 has no triangles. Figures 2, 3, 4 and 5 each have 1, 3, 6 and 10 shaded triangles, respectively. The sequence representing the number of shaded triangles in each figure is 0, 1, 3, 6, 10, ... Recall, that the sequence representing the number of dots in each figure is 1, 3, 6, 10, 15, ... Notice that these two sequences are the same, except the terms are one-off because the first term of the sequence representing the shaded triangles is 0. It follows, then, that in Figure 10 there are $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ shaded triangles. Each shaded triangle has a perimeter of 3. That means the length of all the segments in Figure 10 is $45 \times 3 =$ **135** units.

7. Billy's sequence begins 1/3, 2/9, 3/27, 4/81, ... Let's see if we can employ a technique similar to the one used in the video to determine the sum of all the terms in Billy's sequence, $S = 1/3 + 2/9 + 3/27 + 4/81 + \cdots$. The denominator of each term is a power of 3, so why don't we start by multiplying our sum by 3. We get $3S = 1 + 2/3 + 3/9 + 4/27 + 5/81 + 6/243 + \cdots$. That doesn't give us much, but subtracting we get,

$$\begin{array}{l} 3S = 1 + 2/3 + 3/9 + 4/27 + 5/81 + \cdots \\ \underline{-S = \quad - 1/3 - 2/9 - 3/27 - 4/81 - \cdots} \\ 2S = 1 + 1/3 + 1/9 + 1/27 + 1/81 + \cdots \end{array}$$

Now, all of our numerators are 1, and we have a better chance of being able to cancel some terms. Multiplying that result by 3, we get $6S = 3 + 1 + 1/3 + 1/9 + 1/27 + 1/81 + \cdots$. Once again, subtracting we get,

$$\begin{array}{l} \quad 6S = 3 + 1 + 1/3 + 1/9 + 1/27 + \cdots \\ \underline{-2S = \quad - 1 - 1/3 - 1/9 - 1/27 - 1/81 - \cdots} \\ \quad 4S = 3 \end{array}$$

Now, dividing each side of the equation by 4, we see that $S =$ **3/4**.