MATHCOUNTS Minis — January 2012 Activity Solutions

Warm-Up!

1. We are told that $a = b^2 - 3$. Since b^2 will always be non-negative, we will get the minimum value of a when $b = 0$. Therefore, the smallest possible value of a is $a = 0^2 - 3 = \mathbf{-3}$.

2. (a) $(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1 = \mathbf{x^2 + 2x + 1}$
   (b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = \mathbf{x^2 + 4x + 4}$
   (c) $(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = \mathbf{x^2 + 6x + 9}$

In general, we see that $(x + y)^2 = x^2 + 2xy + y^2$ for any real numbers x and y.

3. Using the above generalization, we have $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + 14x + $ ___. It follows that $2xy = 14x$, and $y = 14x/2x = 7$. The value that completes the square is $y^2 = 7^2 = \mathbf{49}$.

4. We are asked to expand $(a + b + c)^2$. We have $(a + b + c)^2 = (a + b + c)(a + b + c)$. Using the distributive property, we get $a(a + b + c) + b(a + b + c) + c(a + b + c) = a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2 = \mathbf{a^2 + b^2 + c^2 + 2(ab + ac + bc)}$.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

5. We are told that $x + y = 3$ and $x^2 + y^2 = 6$, and then asked to find the value of xy. Squaring both sides of the first equation, we get $(x + y)^2 = 3^2 \rightarrow x^2 + 2xy + y^2 = 9 \rightarrow x^2 + y^2 + 2xy = 9$. We can substitute 6 for $x^2 + y^2$ to get $6 + 2xy = 9 \rightarrow 2xy = 3 \rightarrow xy = \mathbf{3/2}$.

6. Let $x = 20122011$, and let $y = 20122009$. We are asked to evaluate the expression $20122011^2 - 2(20122011)(20122009) + 20122009^2$, which can be rewritten as $x^2 - 2xy + y^2$. Factoring, we see this is equivalent to $(x - y)^2$. So we have $(x - y)^2 = (20122011 - 20122009)^2 = 2^2 = \mathbf{4}$.

7. We are told that $a + (1/a) = 6$ and asked to determine the value of $a^4 + (1/a^4)$. Squaring the first equation yields $(a + (1/a))^2 = 6^2 \rightarrow a^2 + 2 + (1/a^2) = 36 \rightarrow a^2 + (1/a^2) = 34$. Now squaring each side of this equation, we get $(a^2 + (1/a^2))^2 = 34^2 \rightarrow a^4 + 2 + (1/a^4) = 1156$. Therefore, $a^4 + (1/a^4) = \mathbf{1154}$.

8. We are told that $xyz = 45$ and $1/x + 1/y + 1/z = 1/5$. We can rewrite the left side of the second equation using a common denominator to get $(yz/xyz) + (xz/xyz) + (xy/xyz) = 1/5 \rightarrow (xy + xz + yz)/xyz = 1/5$. But we know that $xyz = 45$, so we have $(xy + xz + yz)/45 = 1/5 \rightarrow xy + xz + yz = 9$. If the sum of the three products xy, xz and yz is 9, then their mean is $9/3 = \mathbf{3}$.