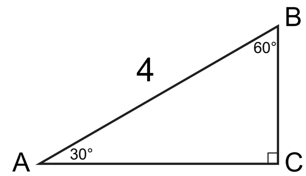
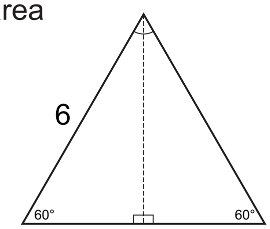
MATHCOUNTS Minis — December 2011 Activity Solutions

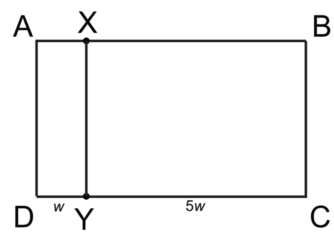
Warm-Up!



1. The first thing we notice is that ΔABC is a 30-60-90 right triangle. This is a special type of right triangle with the following properties: (1) the length of the shorter leg is half the length of the hypotenuse, and (2) the length of the longer leg is equal to $\sqrt{3}$ times the length of the shorter leg. To determine the area of ΔABC we need to find the lengths of the two legs. If $m\angle A = 30°$ and $m\angle B = 60°$, it follows that $m\angle C = 90°$. Since side AB is opposite $\angle C$ it must be the hypotenuse of the triangle. We are told that AB = 4, so the shorter leg (opposite $\angle A$) must have length $(1/2) \times 4 = 2$. The longer leg (opposite $\angle B$) then must have length $2\sqrt{3}$. Therefore, the area of ΔABC is $(1/2)(2)(2\sqrt{3}) = \mathbf{2\sqrt{3}}$.



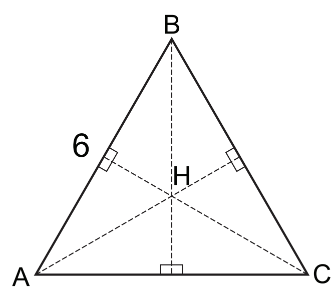
2. We know the base of this equilateral triangle has length 6. To determine the area we need to find the height of the triangle. If we draw the altitude from the vertex angle of the triangle to the opposite side, as shown, it divides the triangle into two 30-60-90 right triangles, each with a hypotenuse of length 6. Based on the properties of 30-60-90 right triangles, the length of the shorter leg of each of the right triangles is $(1/2) \times 6 = 3$. The length of the longer leg, which is the height of the equilateral triangle, then is $3\sqrt{3}$. We now see that the area of this equilateral triangle is $(1/2)(6)(3\sqrt{3}) = \mathbf{9\sqrt{3}}$.



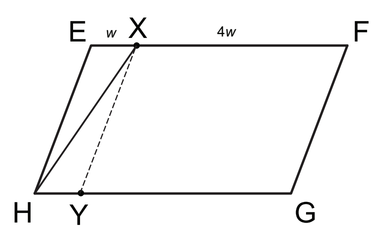
3. We are told that the ratio of the areas of rectangles XBCY and AXYD, shown here, is 5/1. Since both rectangles have the same height, it follows that the ratio of the width of rectangle XBCY to the width of rectangle AXYD is CY/YD = 5/1. If we let YD = w, then CY = $5w$ and CD = $6w$. Therefore, the ratio of YD to CD is $w/6w$ = **1/6**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems

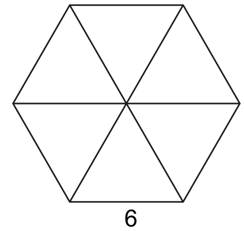


4. Recall from the solution of the third problem in the video that the altitudes of an equilateral triangle, which bisect the angles of the triangle, intersect at a point in the center of the triangle that is equidistant from each side. Drawing these altitudes in ΔABC, as shown, we see that six congruent 30-60-90 right triangles are created. We are asked to find AH, to which we will assign the variable x. Since side AH is the hypotenuse of one of these 30-60-90 right triangles, is follows that the shorter leg has length $x/2$, and the longer leg has length $(x/2)\sqrt{3}$. But we know that the length of each side of ΔABC is 6, and the length of the longer leg of one of the 30-60-90 right triangles is half that amount, $(1/2) \times 6 = 3$. So we have that $(x/2)\sqrt{3} = 3 \rightarrow x/2 = 3/\sqrt{3} \rightarrow x\sqrt{3} = 6 \rightarrow x = 6/\sqrt{3}$. Simplifying, we see that x = AH = $\mathbf{2\sqrt{3}}$.

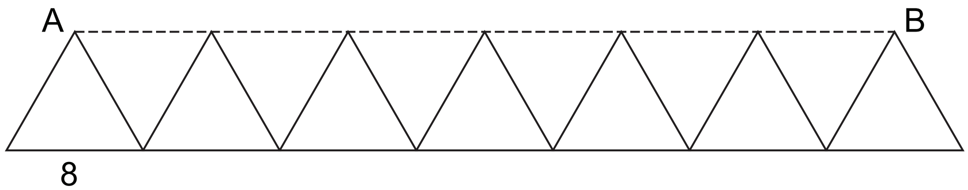


5. We are asked to determine the ratio of the area of ΔEXH to the area of parallelogram EFGH. If we draw a point Y on side GH such that GH = 5YH, parallelogram EXYH is created. The area of ΔEXH is 1/2 the area of parallelogram EXYH. The area of parallelogram EXYH is 1/5 times the area of parallelogram EFGH. Therefore, $\frac{[\Delta EXH]}{[\square EFGH]} = \frac{1}{2} \times \frac{1}{5} = \mathbf{\frac{1}{10}}$.

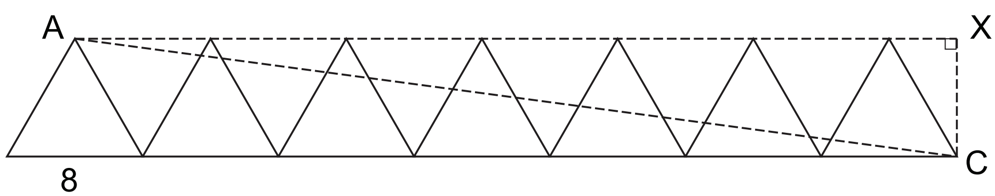
6. Drawing the three diagonals of the regular hexagon, as shown, we see that six congruent equilateral triangles are created, each with sides of length 6. Recall from problem #2 that the area of an equilateral triangle with side length 6 is $2\sqrt{3}$. Since the hexagon is composed of six of these congruent triangles, it follows that the area of the hexagon is $6 \times 2\sqrt{3} = \mathbf{12\sqrt{3}}$.



7. To determine the distance from point A to point B, we can use the technique employed in the third problem of the video. Construct segment AB, as shown, and notice that six new congruent equilateral triangles are created, each with sides of length 8. We can see that the distance from point A to point B is $6 \times 8 =$ **48**.



Extra challenge: If we draw segment AC and then draw an altitude from point C intersecting the line containing segment AB at point X, as shown, we create right triangle AXC. We can see that the length of the shorter leg of ΔAXC has the same length as the altitude of one of the equilateral triangles. Using the properties of 30-60-90 right triangles, we can determine that the length of the altitude is $4\sqrt{3}$ since the hypotenuse of the 30-60-90 right triangle is 8 and the shorter leg has length 4. The length of segment AX is $(6 \times 8) + 4 = 48 + 4 = 52$. Now we can use the Pythagorean Theorem to determine the distance from point A to point C. We have $AC^2 = AX^2 + XC^2 \rightarrow AC = \sqrt{(52^2 + (4\sqrt{3})^2)} = \sqrt{(2704 + 48)} = \sqrt{(2752)} = \mathbf{8\sqrt{43}}$.



8. Let's extend segments AD and BC until they intersect at point E, as shown. Notice that $m\angle EBA = 180 - 120 = 60°$, and $m\angle BAE = 180 - 90 = 90°$. That means the $m\angle E = 30°$, and ΔABE is a 30-60-90 right triangle. We know that AB = 3, so using the properties of 30-60-90 right triangles, we see that $EB = 2 \times 3 = 6$. Now consider right triangle CDE with $m\angle C = 90°$ and $m\angle E = 30°$. It follows that $m\angle D = 60°$ making ΔCDE a 30-60-90 right triangle. The length of the longer leg is $EC = EB + BC = 6 + 4 = 10$. Segment CD is the shorter leg of ΔCDE. Therefore, according to the properties of 30-60-90 right triangles, we have $CD = \frac{CA}{\sqrt{3}} = \frac{10}{\sqrt{3}} = \mathbf{\frac{10\sqrt{3}}{3}}$.