Warm-Up!
1. The area of rectangle ABCD is $27(11) = 297$ sq units. If we subtract the area of the triangular region that is removed from the area of rectangle ABCD, the result is the area of pentagon ABEFD. Now $CF = CD - FD = 27 - 15 = 12$ units, and $EC = BC - BE = 11 - 6 = 5$ units. Thus the area of $\triangle CEF$ is $\frac{1}{2}(12)(5) = 30$ sq units. That means the area of pentagon ABEFD is $297 - 30 = 267$ sq units.

2. We are told that segment XY intersects the center of the square, which means $CY = AX = 2$ units. We can now determine that $DY = DC - YC = 10 - 2 = 8$ units. Now quadrilateral AXYD is a trapezoid with bases $DY$ and $AX$. The area of a trapezoid is just the average of the lengths of its bases times its height. Thus, the area of trapezoid AXYD is $\frac{1}{2}(2 + 8)(10) = \frac{1}{2}(10)(10) = 5(10) = 50$ sq units.

3. We know that the $m\angle APB = m\angle DPC$ because opposite angles are congruent. Also, we know that $m\angle PCD = m\angle PBA$ since alternate interior angles are congruent. From this we conclude that $\triangle CDP \sim \triangle BAP$ and $\frac{CP}{14 - CP} = \frac{8}{20}$. After cross-multiplying, the resulting equation is $CP(20) = (14 - CP)(8) \rightarrow CP(20) = 112 - CP(8) \rightarrow CP(28) = 112 \rightarrow CP = 4$ units.

4. The area of a triangle is determined by taking half the product of the length of its base and its height. The two triangles formed by the median have the same height and the lengths of their bases are equal, therefore the area of the two triangles is equal.

5. In order to determine the area of $\triangle PQR$ we need to know the length of the base and the height. We know $PQ = 3$, and segment PQ is a base of $\triangle PQR$. The height of $\triangle PQR$ is the same as the height of the trapezoid. We are told that the area of trapezoid PQRS is 24 sq units and that the lengths of the bases are 3 units and 9 units. We can use this information to determine the height, $h$, of the trapezoid. We have $\frac{1}{2}(3 + 9)h = 24 \rightarrow \frac{1}{2}(12)h = 24 \rightarrow 6h = 24 \rightarrow h = 4$ units. Now we see that the area of $\triangle PQR$ is $\frac{1}{2}(3)(4) = \frac{1}{2}(12) = 6$ sq units.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems
6. The figure appears to be a rectangle from which side $CD$ has been removed and two of the three interior triangles have been shaded, as shown. If we determine the area of rectangle ABCD and subtract from it the area of the unshaded triangle, the result will be the area of the shaded region. The area of the rectangle is $(8)(12) = 96$ cm$^2$. The area of the unshaded triangle is $\frac{1}{2}(12)(8) = (6)(8) = 48$ cm$^2$. That means the area of the shaded region is $96 - 48 = 48$ cm$^2$.

You may also have recognized that if we rotate the two shaded triangles $180^\circ$ and translate them so that segment BC and segment AD perfectly overlap, the resulting figure fits the unshaded region exactly. The combined area of the two shaded triangles is the same as the area of the unshaded triangle with base = 12 cm and height = 8 cm.
7. To find the area of this irregular-shaped region, let’s first determine the area of the large rectangular region. The area is $40(35) = 1400$ sq ft. Next, we can break up the area outside the irregular shape, but inside the rectangular region, into several shapes with easily determined areas, as shown. The triangle in the upper right corner of the rectangular region has an area of $\frac{1}{2}(10)(15) = 75$ sq ft. The triangle in the lower left corner has an area of $\frac{1}{2}(20)(5) = 50$ sq ft. The area of the trapezoid in the lower right corner is $\frac{1}{2}(5 + 15)(20) = \frac{1}{2}(20)(20) = 200$ sq ft. That means the area of the irregular-shaped region is $1400 - (75 + 50 + 200) = 1400 - 325 = 1075$ sq ft.

8. The area of $\triangle ABC$ is the sum of the areas of $\triangle CAP$ and $\triangle CBP$. We are told that the area of $\triangle CAP$ is 40 sq units. We do not know the length of segment $AP$, so let’s call it $2x$. That means that $\frac{1}{2}(2x)h = 40$. In other words, $xh = 40$. We are told that $AP:PB = 1:2$. If $AP = 2x$ then $PB = 4x$ and the area of $\triangle CBP$ is $\frac{1}{2}(4x)h = 2xh$. We know that $xh = 40$ sq units so, $2xh = 80$ sq units. Now we can see that the area of the $\triangle ABC$ is equal to $40 + 80 = 120$ sq units.

9. No matter what the lengths are for $WX$ and $YZ$, their sum will always be 8 units. The area of trapezoid $WXYZ$ is $\frac{1}{2}(WX + YZ)(8) = \frac{1}{2}(8)(8) = 32$ sq units. Also see that if we tesselate trapezoid $WXYZ$ to create trapezoid $W'X'Y'Z'$, then put the two figures together, as shown, they would form a square. So, no matter what the lengths are of the bases of this trapezoid, as long as the sum of the lengths is 8 units, the area of the trapezoid will 32 sq units, half the area of the square.

10. Harvey’s diagram is a tessellation of the plane with copies of the original trapezoid. We are asked for the area of one blue triangle. In the tessellation, the diagonals of the trapezoids line up conveniently to form parallel strips in two directions. In each direction, the strip’s widths have ratio 3:2 (from the bases of the trapezoid). Each color corresponds to one possible pairing of intersections of types of strips. For example, the intersection of the wide strips that go up from left to right with the narrow strips that go down from left to right are entirely blue. Those that are intersections of narrow strips are yellow, and so on. So, when we tile the whole plane with these colored trapezoids, the blue trapezoids are $2/5$ of $3/5$ of the whole plane. Each trapezoid is colored in the same ratios, so the blue of each trapezoid is $2/5$ of $3/5$ of each trapezoid.
Follow-up Problems
11. The height, \( h_1 \), of \( \triangle APF \) is equivalent to that of \( \triangle BPF \). Using the notation from the video we have 
\[
\frac{\triangle APF}{\triangle BPF} = \frac{\frac{1}{2}AF(h_1)}{\frac{1}{2}BF(h_1)} = \frac{AF}{BF}.
\]

12. Similarly, the height, \( h_2 \), of \( \triangle ACF \) is equivalent to that of \( \triangle BCF \), and 
\[
\frac{\triangle ACF}{\triangle BCF} = \frac{\frac{1}{2}AF(h_2)}{\frac{1}{2}BF(h_2)} = \frac{AF}{BF}.
\]

13. Let \( x = \frac{1}{2}(h_1) \), and let \( y = \frac{1}{2}(h_2) \). It follows that 
\[
\frac{\triangle ACP}{\triangle BCP} = \frac{\triangle ACP - \triangle APF}{\triangle BCP - \triangle BPF} = \frac{xAF - yAF}{xFB - yFB} = \frac{(x - y)AF}{(x - y)FB} = \frac{AF}{FB}.
\]

14. As in problems 11 through 13, we can show that 
\[
\frac{\triangle ABP}{\triangle ACP} = \frac{BD}{DC} \quad \text{and} \quad \frac{\triangle ABD}{\triangle ACD} = \frac{BD}{DC}
\]
which means that 
\[
\frac{\triangle AABP}{\triangle AACP} = \frac{BD}{DC}.
\]
Likewise, since 
\[
\frac{\triangle CBE}{\triangle ABE} = \frac{CE}{EA} \quad \text{and} \quad \frac{\triangle CPE}{\triangle APE} = \frac{CE}{EA},
\]
we have that 
\[
\frac{\triangle BCP}{\triangle ACP} = \frac{CE}{EA}.
\]
It follows that 
\[
\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{\triangle ACP}{\triangle ACP} \cdot \frac{\triangle ABP}{\triangle ACP} \cdot \frac{\triangle BCP}{\triangle ACP} = 1.
\]