Warm-Up!
1. The arithmetic sequence of the first 31 positive integers consists of the 31 integers 1, 2, 3, ..., 30, 31. There will be the first 15 terms, then the 16th term, which is 16, and then the last 15 terms.

2. An arithmetic sequence with 21 terms (which is an odd number of terms), will have an exact middle term. There will be the first $21 ÷ 2 = 10.5 → 10$ terms, then the 11th term will be the middle term, and then there will be the last 10 terms. The common difference of this arithmetic sequence is 4. To get from the 1st term, 3, to the 11th term, we will need to add the common difference 10 times: $3 + 10(4) = 43$.

3. The common difference of this arithmetic sequence is $33 – 31 = 2$, since 31 and 33 are consecutive terms of the sequence. Because this sequence has an even number of terms, 14, there will be the first $14 ÷ 2 = 7$ terms and the second 7 terms. Therefore, 31 is the 7th term and 33 is the 8th term. To get from the 7th term to the 14th term, we need to add the common difference 7 times: $31 + 7(2) = 45$.

4. The mean of the first 50 positive, even integers will be the sum of the integers divided by 50. This sequence is 2, 4, 6, ..., 98, 100. If we look at the integer pairs, moving from the outside members of the list of 50 integers toward the middle, we have 2 and 100; 4 and 98; 6 and 96; ..., 50 and 52. Notice that the sum of each pair is 102 and there are 25 pairs. Their sum is then $(102)(25) = 2550$. Dividing this by 50 gives us a mean of 51. (Notice that there is no middle term for this arithmetic sequence, but our answer is equal to the mean of the two middle-most terms.)

5. As we found in the last problem, the middle two numbers are 50 and 52.

6a. The average score of the team members’ Sprint Round scores is $(7 + 8 + 21 + 24) ÷ 4 = 60 ÷ 4 = 15$.

6b. The boys’ scores were 8 points and 7 points below the team average, for a total of 15 points.

6c. The girls’ scores were 6 points and 9 points above the team average, for a total of 15 points.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems
7. If the sum of 7 consecutive odd integers is 273, and the middle term is the average of those integers, the middle term is $273 ÷ 7 = 39$. Since these are consecutive odd integers, their common difference is 2. So going from the middle term (or 4th term) to the 7th term, we need to add the common difference 3 times: $39 + 3(2) = 45$. 

© 2011 MATHCOUNTS Foundation. All rights reserved. MATHCOUNTS Mini Solution Set
8. If the sum of the first five terms of an arithmetic sequence is 75, then the middle term, or 3rd term, is $75 \div 5 = 15$. The extended sequence with 11 terms must have a middle term, or 6th term, of $363 \div 11 = 33$. The difference between the 6th term and 3rd term is $33 - 15 = 18$, which is the common difference added on 3 times. So the common difference is $18 \div 3 = 6$. Going from the 3rd term to the 1st term would mean taking the common difference off twice: $15 - 2(6) = 3$.

9. The middle term of the first 9 consecutive odd integers will be the 5th odd integer, which is 9. This is also the mean of the first 9 consecutive odd integers, so their sum is $9 \times 9 = 81$. This is 3 more than the sum of 6 consecutive even integers. Their sum is then 78. Their average value is $78 \div 6 = 13$. So the 3rd and 4th terms are 14 and 16, and the 6th term is 20.

10. If we assume there will only be two consecutive integers with this sum, then we know $210 \div 2 = 105$ is the value *between* them. However, it’s impossible for 105 to be *between* two consecutive integers. What if we assume we can find three consecutive positive integers with a sum of 210? Then the middle term - and with an odd number of terms, there is an exact middle term - will be $210 \div 3 = 70$, and the terms are 69, 70 and 71, with 71 being the largest. This shows us that with an odd number of terms, that odd number must be a factor of 210 in order to get an exact middle term that is an integer. The odd factors of 210 are 1, 3, 5, 7, 15, 21, 35 and 105. If there were 105 terms, then the middle (53rd) term would be $210 \div 105 = 2$, and we can see that spreading out 52 integers in both directions will result in terms that are negative integers. What if there were 35 terms? The middle (18th) term would be $210 \div 35 = 6$, and again we would get terms that are negative integers when spreading out 17 integers in both directions. Similarly, assuming there are 21 terms will result in a middle (11th) term of 10 and a smallest term of 0. However, with 15 terms, the middle (8th) term is $210 \div 15 = 14$, the smallest term is 7 and the largest term is 21. This takes care of the possibilities with an odd number of terms. However, just because 2 terms didn’t work, doesn’t mean there couldn’t be an even number of terms...

Perhaps there can be four terms... $210 \div 4 = 52.5$. Being that four terms is an even number of terms, this 52.5 must be *exactly between* two consecutive integers, and it is: 52 and 53. The set would be 51, 52, 53 and 54, with 54 being the largest. As we can see, this even number cannot be a factor of 21, or it will result in a mean which is an integer and can’t be *between* two consecutive integers. In fact, the result of dividing 210 by this even number must result in a number that is halfway between two consecutive integers. Is there an even number larger than 15 that works (since we know 15 is the largest we’ve come up with so far)? After seeing that 16 and 18 don’t work, we see 20 does. Notice $210 \div 20 = 10.5$, meaning that the 10th term is 10, the 11th term is 11, the first term is 1 (positive) and the last term is 20. If we try any larger number of even terms, we would run into the problem of getting smaller middle values and extending into the negative integers. The answer then is 20 which results from 20 consecutive integers starting with 1.

11. The sum of the interior angles of an octagon is 1080 degrees. Being that there are 8 angles, the average/middle value must be $1080 \div 8 = 135$. Given that the problem says this is an increasing sequence, we know we can’t have an equiangular octagon with 8 angles each measuring 135 degrees. But we have: __ __ __ __ __ __ __ __ (135).
As we saw in the video, if our 4th term is smaller than the average by 1, the common
difference is 2 and we need to ensure the 4th term minus three of these differences is positive.
Additionally, we need to ensure the 5th term plus three of these differences is less than 180 (to
keep the figure convex). This upper limit is what we need to worry about (since 135 is closer to
180 than 0). We see that having 4th and 5th terms of 134 and 136; 133 and 137; and so on will
work up to a certain point. Where is that point? If we add x to 135 to get the 5th angle, then the
common difference of the sequence is 2x. Then the 6th angle is 135 + 3x, the 7th angle is
135 + 5x and the 8th angle is 135 + 7x (which we see is also the 5th angle plus 3 of these
differences of 2x). We need 135 + 7x < 180. Subtracting 135 from both sides and then dividing
by 7 yields x < 6.43..., so x = 6 is the largest value that works. Remember, this was the amount
to add to 135 to get the 5th angle. Therefore, we could add 1, 2, 3, 4, 5 or 6, and there must be
6 possible sequences.

12. Assume each of your first 6 scores were 183. We would need enough “extra points” to
raise each of those games by 7 points (which is a total of 42 points) as well get the 190 points
necessary for the 7th game to maintain the average of 190. That means the 7th game’s score
would need to be 190 + 42 = 232 points.

13. For the original set of 10 numbers, let’s let the average value be x. This tells us the sum of
the numbers is 10x. Removing the largest number results in an average of x −1 and a sum of
9(x − 1) = 9x − 9 for the 9 remaining numbers. The largest number is then 10x − (9x − 9) =
x + 9. Removing the smallest number results in an average of x + 2 and a sum of 9(x + 2) =
9x + 18. The smallest number is then 10x − (9x + 18) = x − 18. The positive difference between
these two numbers is (x + 9) − (x − 18) = x + 9 − x + 18 = 27.