MATHCOUNTS Minis **March 2011 Activity Solutions**

Warm-Up!
1a. To find the surface area we need to sum the areas of all eight faces of the box. Let's label the edge lengths as L = 5 cm, W = 7 cm and H = 8 cm. Since opposite faces are the same we will calculate $2(LW + WH + LH) = 2(5 \times 7 + 7 \times 8 + 5 \times 8) = 2(131) =$ **262 cm²**.

1b. The volume of the box is simply $5 \times 7 \times 8 =$ **280 cm³**.

2. A rectangular box has three dimensions we'll call them X, Y and Z. There are 4 edges with length = X, 4 edges with length = Y and 4 edges with length = Z. That means the total number of edges on a rectangular box is **12**.

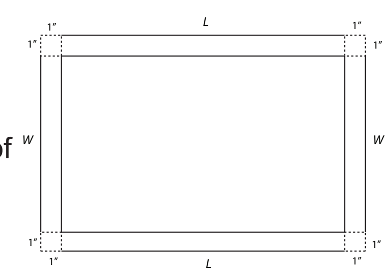
3. If we expand the product $(x + 1)(y + 1)$ using the F.O.I.L. method we get $xy + x + y + 1$. Let's see how this relates to the figure. According to the figure the rectangle with dimensions x and y has an area of xy sq units. Above this rectangle is a smaller 1 by x rectangle, which is shown bounded on the left and above by a solid line, and bounded on the right and below by a dashed line. The area of this rectangle is $1 \times x = x$ sq units. Similarly, the 1 by y rectangle to the right of the original rectangle which is shown bounded to the left and above by a dashed line, and bounded on the right and below by a solid line, has an area of $1 \times y = y$ sq units. That just leaves the 1 by 1 square in the upper right corner of the figure which has an area of $1 \times 1 =$ 1 sq unit. So, it appears that $xy + x + y + 1$ is equal to the sum of the areas of the four regions shown in the figure. In other words, the expansion of the product $(x + 1)(y + 1)$ is **the total area of the rectangle** in the figure.

4. Since we don't know the dimensions of the rectangle let's call them L and W. We are told that the rectangle has an area of 108 in^2 which means that $LW = 108$. We are looking for the area if the length and width are each increased by 1. In other words, area = $(L + 1)(W + 1)$. If we expand this expression we get $LW + L + W + 1$. Well we know that $LW = 108$. We are told that the perimeter of the rectangle is 42 which means that $2(L + W) = 42 \rightarrow L + W = 21$. Substituting, we now have $LW + (L + W) + 1 = 108 + 21 + 1 =$ **130 in²**.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems
5. Let the dimensions of the rectangle be L and W. We know that the area of the rectangle is $LW = 97$ in^2 and the perimeter of the rectangle is $2(L + W) = 44$ in. We can rewrite the second equation as L + W = 22. Using Richard's approach from the video, we see that increasing the length and width by one inch each adds a 1 by L region, a 1 by W region and a 1 by 1 region. The area of those regions is $1 \times L = L$, $1 \times W = W$ and $1 \times 1 = 1$. Thus, the original area of 97 in^2 is increased by $L + W + 1$. We stated earlier that $L + W = 22$. Therefore, the area of the resulting rectangle is $97 + 22 + 1 =$ **120 in²**.

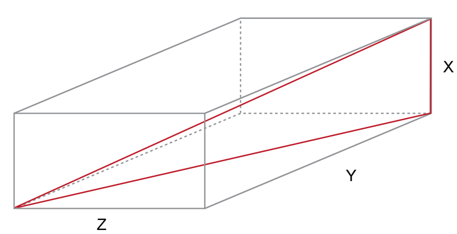
6a. We are told that the perimeter of the painting is 48 in. Since adding a frame that produces a one-inch margin around the painting essentially adds an additional 2 inches at each corner of the painting, the outer perimeter of the frame is 48 + 8 = **56 in**.



6b. We are told that the perimeter of the painting is 48 in. That means $2L + 2W = 48$. As the figure shows, the area of the frame is the sum of the areas of the $1\times L$ regions at the top and bottom of the painting, the $1\times W$ regions on either side and the 1×1 regions at each of the four corners. Thus, the area is of the frame is $2L + 2W + 4 = 48 + 4 =$ **52 in²**.

7. Let's label the lengths of the edges of the box as X, Y and Z. The problem states that the areas of the three faces of the rectangular box are 18 ft², 75 ft² and 24 ft². That means that $XY = 18$, $YZ = 75$ and $XZ = 24$. We don't know the value of X, Y or Z so we can't just calculate $V = XYZ$ this time. If we multiply the three areas we get $(XY)(YZ)(XZ) = (XYZ)^2 = V^2 = (18)(75)(24) = 32{,}400$. That means that the volume is equal to the positive square root of 32,400. Thus, V = **180 ft³**.

8. We shall, again, call the length, width and height of the original box in the video X, Y and Z. If X, Y and Z were each increased by 2 inches the volume would then be equal to $(X + 2)(Y + 2)(Z + 2)$. Expanding this yields $XYZ + 2(XZ + YZ + XY) + 4(X + Y + Z) + 8$. Let's see if we can identify some known values in this expression. We know that $XYZ = 4{,}320$, the volume of the orignal box. We also know that $2(XZ + YZ + XY) = 1{,}704$, the surface area of the original box. Finally, the sum of the edges of the original box is $4(X + Y + Z) = 208$. So what does the 8 represent? Remember in the video after Richard added the 1-inch regions to each face of the box and to each of the edges, he had to add the 1" × 1" × 1" cube at the end. This time we've added 2 inches to the length, width and height so, we've added a 2" × 2" × 2" cube. The volume of this cube is 8 in³. Therefore, the volume of the resulting box is 4,320 + 1,704 + 208 + 8 = **6,240 in³**.



9. We are told that the sum of the edges of a rectangular box is 92 cm. We'll continue to label the length, width and height as X, Y and Z. That means that $4(X + Y + Z) = 92 \rightarrow X + Y + Z = 23$. As the figure shows, the interior diagonal of the box is the hypotenuse of a right triangle. The short leg of this triangle is simply the measure of one of the edges with length X. The long leg of this right triangle is, in fact, the hypotenuse of a different right triangle formed with two other edges of length Y and Z as legs. Therefore, the length of the long leg of the right triangle pictured is equal to $\sqrt{Y^2+Z^2}$. So, with legs of length X and $\sqrt{Y^2+Z^2}$, the length of the interior diagonal can be expressed as $\sqrt{\left(\sqrt{Y^2+Z^2}\right)^2 + X^2} = \sqrt{X^2+Y^2+Z^2}$. Since we don't know the value of X, Y and Z, we are unable to just substitute these values into the equation. It appears we need to find the square of X, Y and Z, or at least $X^2 + Y^2 + Z^2$. Let's try squaring each side of the equation $X + Y + Z = 23$. The result is $(X + Y + Z)(X + Y + Z) = 23^2 \rightarrow X^2 + XY + XZ + Y^2 + YZ + XY + Z^2 + XZ + YZ = 529 \rightarrow X^2 + Y^2 + Z^2 + 2(XY + XZ + YZ) = 529$. Since the surface area of the box is 240 cm² we can substitute 240 for the expression $2(XY + XZ + YZ)$ in the equation to get $X^2 + Y^2 + Z^2 + 240 = 529 \rightarrow X^2 + Y^2 + Z^2 = 289$. Therefore, the length of the interior diagonal is $\sqrt{X^2+Y^2+Z^2} = \sqrt{289} =$ **17 cm**.