MATHCOUNTS Minis September 2010 Activity Solutions

Warm-Up!

1a. Solve by substituting 3 for x. So, $y = 2(3) - 7 \rightarrow y = 6 - 7 \rightarrow y = \mathbf{-1}$.
1b. Solve by substituting 7 for x. So, $y = \frac{2(7-3)}{7+1} \rightarrow y = \frac{2(4)}{8} \rightarrow y = \frac{8}{8} \rightarrow y = \mathbf{1}$.
1c. Solve by substituting $y + 2$ for x. So, $y = 2(y + 2) - 3 \rightarrow y = 2y + 4 - 3 \rightarrow y = 2y + 1$. Then subtract $2y$ from each side to isolate the variable and combine like terms to get $-y = 1$. Finally, divide each side by −1 to get your final answer $y = \mathbf{-1}$.

2a. Using the F.O.I.L. method, find the sum of the product of the first terms, the product of the outside terms, the product of the inside terms, and the product of the last terms: $(x + 1)^2 = (x + 1)(x + 1) = x^2 + x + x + 1$. Now combine like terms to get the result $\mathbf{x^2 + 2x + 1}$.
2b. Again using F.O.I.L., $(t + 3)^2 = (t + 3)(t + 3) = t^2 + 3t + 3t + 9$. Combine like terms to get $\mathbf{t^2 + 6t + 9}$.
2c. Again using F.O.I.L., $(y - 1)^2 = (y - 1)(y - 1) = y^2 - y - y + 1$. Combine like terms to get $\mathbf{y^2 - 2y + 1}$.

3a. Since $c^2 + 2c + 1 = (c + 1)(c + 1) = (c + 1)^2$, **yes**, $c^2 + 2c + 1$ is the square of a binomial.
3b. Since $u^2 - 4u + 3 = (u - 1)(u - 3)$, **no**, $u^2 - 4u + 3$ is not the square of a binomial.
3c. Since $a^2 + 6a + 9 = (a + 3)(a + 3) = (a + 3)^2$, **yes**, $a^2 + 6a + 9$ is the square of a binomial.

4. Let l = the total number of dolls Linda has, and let s = the total number of dolls Sarah has. Then $l = 2s - 3$ and $l + s = 60$. Solving the second equation for l we get $l = 60 - s$. Now, substitute $60 - s$ for l in the first equation to get $60 - s = 2s - 3$. Next add s to each side to isolate the variable and divide each side by 3, to get $63 - s = 2s \rightarrow 63 = 3s \rightarrow 21 = s$. Sarah has **21** porcelain dolls.

5. Let $x = 0.\overline{7}$, then $10x = 7.\overline{7}$. That means $10x = 7 + 0.\overline{7} = 7 + x$. If $10x = 7 + x$ and we subtract x from each side of the equation we are left with $9x = 7$. Then dividing each side of the equation by 9 and solving for x we obtain the solution $x = \mathbf{7/9}$.

6. First, isolate the radical expression on one side of the equation $\sqrt{5-x} - 5 = 9 \rightarrow \sqrt{5-x} = 14$. Next, square each side to obtain $5 - x = 196$. Finally, isolating the variable and solving for x we get $-x = 191 \rightarrow x = \mathbf{-191}$.

7. Using F.O.I.L. multiply the two binomials on one side of the equation and combine like terms: $\frac{10}{3}a - \frac{4}{9}a^2 - 25 + \frac{10}{3}a = -81 \rightarrow -\frac{4}{9}a^2 + \frac{20}{3}a - 25 = -81$. Add 81 to each side to get $-\frac{4}{9}a^2 + \frac{20}{3}a + 56 = 0$. Next, multiply each side of the equation by 9 to eliminate the fractions and obtain $-4a^2 + 60a + 504 = 0$. To get a leading coefficient of 1 divide both sides of the equation by −4 to get $a^2 - 15a - 126$. Factoring the quadratic we get $(a - 21)(a + 6) = 0$. That means $a - 21 = 0$ or $a + 6 = 0$. Solving each of these equations for a, we find that $a = \mathbf{21}$ or $a = \mathbf{-6}$.

Consider a simpler solution, if you recognize that $(\frac{2}{3}a - 5)$ and $(5 - \frac{2}{3}a)$ are opposites. That is $(5 - \frac{2}{3}a) = -(\frac{2}{3}a - 5)$. Therefore the original equation can be rewritten as $-(\frac{2}{3}a - 5)^2 = -81$ or $(\frac{2}{3}a - 5)^2 = 81$. Taking the square root of each side yields $\frac{2}{3}a - 5 = 9$ or $\frac{2}{3}a - 5 = -9$. Solving for a gives us $a = \mathbf{21}$ or $a = \mathbf{-6}$.

The Problem is solved during the MATHCOUNTS Mini.

Follow-up Problems

8. Let $\sqrt{90+\sqrt{90+\sqrt{90+\ldots}}}$ = x. Therefore, squaring each side gives us $90+\sqrt{90+\sqrt{90+\ldots}}=x^2$. But since $\sqrt{90+\sqrt{90+\ldots}}=x$, we can substitute to get $90 + x = x^2$. Subtract 90 and x from each side to set the quadratic equal to 0. So, $0 = x^2 - x - 90$, and factoring the quadratic we get $0 = (x + 9)(x - 10) = 0$. That means $x + 9 = 0$ or $x - 10 = 0$. Solving each of these equations for x, we find that $x = -9$ or $x = 10$. However, since the square root of a number is always greater than or equal to 0 we discard $x = -9$ as an extraneous solution.
Therefore, $\sqrt{90+\sqrt{90+\sqrt{90+\ldots}}}$ = **10**. Testing your answer with the calculator you find that each successive result gets closer to 10.

9. Using the technique demonstrated in the Mini video, we start by squaring each side of the equation to eliminate a radical. So, $x-\sqrt{x-\sqrt{x-\ldots}}=16$. But since $\sqrt{x-\sqrt{x-\ldots}}=4$, we can substitute to get $x - 4 = 16$. Adding 4 to each side of the equation gives us x = **20**.

10. Let $x = 0.00\overline{6}$. Then $10x = 0.06\overline{6}$. Consider $10x - x = 0.06\overline{6} - 0.00\overline{6} \rightarrow 9x = 0.06$ or $900x = 6$. This leads to $x = 6/900$ = **1/150**.

11. If $a = 3$, then $c = a + b = 3 + b$. We then can write $d = b + c$ as $d = b + (3 + b) = 3 + 2b$. We then can write $e = c + d$ as $e = (3 + b) + (3 + 2b) = 6 + 3b$. We can write $f = d + e$ as $49 = (3 + 2b) + (6 + 3b) \rightarrow 49 = 9 + 5b \rightarrow 40 = 5b \rightarrow b = 8$. Substituting $b = 8$ into $e = 6 + 3b$ gives us $e = 6 + 3(8) = 30$. So $g = e + f$ is $g = 30 + 49$ = **79**.

12. Let $1+\frac{6}{1+\frac{6}{1+\frac{6}{1+\ldots}}}$ = x. That means $1 + \frac{6}{x} = x$. To solve for x, first multiply each side of the equation by x to eliminate the fraction. We have $x + 6 = x^2 \rightarrow 0 = x^2 - x - 6$. Factoring the quadratic we get $0 = (x + 2)(x - 3)$. That means $0 = x + 2 \rightarrow x = -2$, or $0 = x - 3 \rightarrow x = 3$. We discard $x = -2$ since $x > 0$, so the solution is x = **3**.

13. Given that a and b are both positive and assuming $\frac{a+b}{2} \geq \sqrt{ab}$, if we multiply each side of the equation by 2 we have $a + b \geq 2\sqrt{ab}$. Now square each side of the equation to eliminate the radical to get $(a + b)^2 \geq 4ab$. Squaring the binomial gives us $a^2 + ab + ab + b^2 \geq 4ab \rightarrow a^2 + 2ab + b^2 \geq 4ab$. Now subtract 4ab from each side of the equation and get $a^2 + 2ab + b^2 - 4ab \geq 0 \rightarrow a^2 - 2ab + b^2 \geq 0$. If we factor the quadratic we get $(a - b)(a - b) \geq 0 \rightarrow (a - b)^2 \geq 0$. This is true, since the square of any quantity is always greater than or equal to 0.