Number Lines & Coordinate Planes Meeting (Multiple Topics)

**Topic**
Through the use of number lines and coordinate planes, topics such as area, midpoints, proportional reasoning, distance formula and rotations are covered in the problem set.

**Materials Needed**
- Copies of the Number Lines & Coordinate Planes problem set (Problems and answers can be viewed below. Complete solutions and a more student-friendly version of the problems—with pictures and larger font—are available for download from www.mathcounts.org on the MCP Members Only page of the Club Program section.)
- Calculators
- Graph Paper - this is not required, but may be beneficial to some students

**Meeting Plan**
The problems in this problem set increase in difficulty as students progress. Given the wide range of topics that are covered, it is suggested that students work on the problems in pairs or small groups. Students who are taking or have completed algebra and/or geometry should be paired with students who have not.

1. What is the coordinate of the point located one-half of the way from –1/2 to 5/4 on a number line? Express your answer as a common fraction. 2006 School Competition Sprint Round #25

2. On a number line, the coordinates of P and Q are 8 and 48, respectively. The midpoint of segment PQ is B, the midpoint of segment BQ is C, and the midpoint of segment PC is D. What is the coordinate of D? 2004-2005 School Handbook Warm-Up 3-4

3. Nine points are to be evenly spaced on a standard number line. They are labeled A through I, from left to right. The coordinates of points A and I are 1 and 10, respectively. What is the coordinate of point B? Express your answer as a mixed number. 2005-2006 School Handbook Warm-Up 17-9

4. What is the midpoint of the segment connecting (3, 4) and (3.8, 5.6) in the Cartesian plane? Express the coordinates as decimals to the nearest tenth. 2006-2007 School Handbook Warm-Up 4-7

5. Point C is on the segment AB which has endpoints A(2, –1) and B(11, 5). Point C is twice as far from point A as it is from point B. What are the coordinates of point C? 2007 School Competition Sprint Round #21

6. In the coordinate plane, what is the distance between the point with coordinates (3, 5) and the point with coordinates (–5, 20)? 2006-2007 School Handbook Warm-Up 4-7

7. Using the coordinate plane, follow the clues to find the location of the treasure.
   - Start at (–1, –1). Follow the line y = x until x = 4.
   - Reflect this new point over the x-axis.
   - From this new point add seven to the y-coordinate and subtract five from the x-coordinate.
   This is the location of the treasure. How far is the treasure point from the starting point? 2005-2006 School Handbook Warm-Up 4-1

8. What is the area of the circle that is centered at the origin and is tangent to the line y = 7? Express your answer in terms of π. 2007-2008 School Handbook Warm-Up 3-1
9. The vertices of triangle ABC are A(1, –2), B(4, 0) and C(2, 2). What is the area of triangle ABC? 2005-2006 School Handbook Workout 5-3

10. What is the area of the convex quadrilateral with vertices (–1, 0), (0, 1), (2, 0) and (0, –3)? 2007 School Competition Sprint Round #20

11. The rectangular region bounded by the lines with equations $x = 1.2$, $x = 2.6$, $y = –0.2$ and $y = d$ has area 14 square units. What is the greatest possible value of $d$? Express your answer as a decimal to the nearest tenth. 2002-2003 School Handbook Warm-Up 5-10

12. Three of the four vertices of a rectangle are (5, 11), (1, 11) and (1, –2). What is the area of the intersection of this rectangular region and the region inside the graph of the equation $(x – 5)^2 + (y + 2)^2 = 9$? Express your answer as a common fraction in terms of $\pi$. 2006-2007 School Handbook Warm-Up 13-9

13. Circles with centers at (2, 2) and (17, 10) are both tangent to the $x$-axis. What is the distance between the closest points of the two circles? 2007-2008 School Handbook Warm-Up 6-10

14. When the point (4, 1) is rotated 90 degrees counterclockwise about the point (1, 0), what are the coordinates of the point at which it lands? 2006-2007 School Handbook Warm-Up 15-2

Answers: 3/8; 23; 2 1/8; (3.4, 4.8); (8, 3); 17 units; 4 units; 49 $\pi$ units$^2$; 5 units$^2$; 6 units$^2$; 9.8; $(9\pi)/4$ units$^2$; 5 units; (0, 3)

Possible Next Step
Problems such as #10 can be solved with something known as the Shoestring Algorithm. This algorithm will enable students to find the area of any convex polygon if the coordinates of each vertex is known.

The region for #10 is shown to the right. First, write the coordinates of the vertices in a column. Any vertex can be listed first. The other vertices must be listed in order when going around the polygon either clockwise or counterclockwise. The coordinates of the first vertex are then written again at the bottom of the column. For #10, this could take many forms:

\[
\begin{align*}
C & (0, -3) & B & (2, 0) & A & (0, 1) \\
D & (1, 0) & A & (0, 1) & B & (2, 0) \\
E & (0, 1) & D & (1, 0) & C & (0, -3) \\
F & (2, 0) & C & (0, -3) & D & (1, 0) \\
G & (0, -3) & B & (2, 0) & A & (0, 1)
\end{align*}
\]

Now, multiply the pairs of numbers down the diagonals shown (dotted lines) and write the products off to the side. Then add each column of products down each side of the column of coordinates. Finally, take the positive difference of the two sums and then divide by 2. This is the area of the convex polygon formed by the points with these coordinates.

\[
\begin{align*}
\left| 5 - (-7) \right| &= 12 & \left| -7 - 5 \right| &= 12 & \left| 5 - (-7) \right| &= 12 \\
12 + 2 &= 14 \text{ units}^2 & 12 + 2 &= 14 \text{ units}^2 & 12 + 2 &= 14 \text{ units}^2
\end{align*}
\]
1. ___________  What is the coordinate of the point located one-half of the way from $-\frac{1}{2}$ to $\frac{5}{4}$ on a number line? Express your answer as a common fraction.

2. ___________  On a number line, the coordinates of P and Q are 8 and 48, respectively. The midpoint of segment PQ is B, the midpoint of segment BQ is C, and the midpoint of segment PC is D. What is the coordinate of D?

3. ___________  Nine points are to be evenly spaced on a standard number line. They are labeled A through I, from left to right. The coordinates of points A and I are 1 and 10, respectively. What is the coordinate of point B? Express your answer as a mixed number.

4. (       ,       ) What is the midpoint of the segment connecting (3, 4) and (3.8, 5.6) in the Cartesian plane? Express the coordinates as decimals to the nearest tenth.

5. (       ,       ) Point C is on the segment AB which has endpoints A(2, −1) and B(11, 5). Point C is twice as far from point A as it is from point B. What are the coordinates of point C?

6. _________ units  In the coordinate plane, what is the distance between the point with coordinates (3, 5) and the point with coordinates (−5, 20)?

7. _________ units  Using the coordinate plane, follow the clues to find the location of the treasure.
   • Start at (−1, −1). Follow the line $y = x$ until $x = 4$.
   • Reflect this new point over the x-axis.
   • From this new point add seven to the y-coordinate and subtract five from the x-coordinate.
This is the location of the treasure. How far is the treasure point from the starting point?
8. _______ units$^2$ What is the area of the circle that is centered at the origin and is tangent to the line $y = 7$? Express your answer in terms of $\pi$.

9. _______ units$^2$ The vertices of triangle ABC are A(1, −2), B(4, 0) and C(2, 2). What is the area of triangle ABC?

10. _______ units$^2$ What is the area of the convex quadrilateral with vertices (−1, 0), (0, 1), (2, 0) and (0, −3)?

11. _______ The rectangular region bounded by the lines with equations $x = 1.2$, $x = 2.6$, $y = −0.2$ and $y = d$ has area 14 square units. What is the greatest possible value of $d$? Express your answer as a decimal to the nearest tenth.

12. _______ units$^2$ Three of the four vertices of a rectangle are (5, 11), (16, 11) and (16, −2). What is the area of the intersection of this rectangular region and the region inside the graph of the equation $(x − 5)^2 + (y + 2)^2 = 9$? Express your answer as a common fraction in terms of $\pi$.

13. _______ units Circles with centers at (2, 2) and (17, 10) are both tangent to the $x$-axis. What is the distance between the closest points of the two circles?

14. ( , ) When the point (4, 1) is rotated 90 degrees counterclockwise about the point (1, 0), what are the coordinates of the point at which it lands?
Problem 1. We could simply find the average of these two coordinates on the number line to find their midpoint. Doing this, we get \([(-1/2) + (5/4)] ÷ 2 = [3/4] ÷ 2 = 3/8\). However, to visualize this, students can determine the distance from -1/2 to 5/4 on the number line is 1/2 + 5/4 = 7/4 (since if we start at -1/2, it is 1/2 units to 0 and then another 5/4 units to 5/4). Half of this distance is (7/4) ÷ 2 = 7/8. If we move 7/8 units left from 5/4 (which is 10/8), we end at 10/8 – 7/8 = 3/8.

Problem 2. The distance between P and Q is 48 – 8 = 40. The midpoint of segment PQ is halfway between them, or 20 units to the right of P. This point, B, is thus 8 + 20 = 28. Similarly, point C is halfway between 28 and 48 at 38. Point D is halfway between 8 and 38 at 23.

Problem 3. Two of the nine points already have been placed at 1 and 10. This means the other seven points must be placed to split the nine units between A and I into eight equal sections. Each point must be 9 ÷ 8 = 1 1/8 units from its nearest neighbor(s), so point B is 1 1/8 units from point A and is located at 1 + 1 1/8 = 2 1/8.

Problem 4. Moving from (3, 4) to (3.8, 5.6) requires us to move 0.8 units to the right and 1.6 units up. Moving half of this distance would require moving 0.4 units to the right and 0.8 units up from (3, 4). This puts us at the midpoint (3 + 0.4, 4 + 0.8) = (3.4, 4.8).

Problem 5. In order to move from point A to point B, one has to move 9 units right and 6 units up. Since point C is between A and B and is twice as far from A as it is from B, C must divide line AB in a ratio of 2:1. This means we can think of there being “3 parts” to segment AB and each “part” is 9 ÷ 3 = 3 units right and 6 ÷ 3 = 2 units up when starting at A. Moving two of the “parts” from A (2, –1), we end at (2 + 3 + 3, –1 + 2 + 2) = (8, 3). Notice that moving one more “part” would get us to B since (8 + 3, 3 + 2) = (11, 5).

Problem 6. We could certainly apply the distance formula to the two ordered pairs to get our answer. However, if students don’t know that, they can graph the line segment. Extending a vertical segment down from (-5, 20) and a horizontal to the left from (3, 5), one would see that these segments intersect at (-5, 5) and a triangular region is formed. The vertical leg measures 15 units, the horizontal leg measures 8 units and the remaining segment is our hypotenuse. The Pythagorean Theorem can be applied or students may see that it is the hypotenuse of an 8-15-17 Pythagorean Triple. The length we were looking for is 17 units.

Problem 7. We will keep track of the coordinates at each step. Start at (-1, -1). Follow the line \(y = x\) until \(x = 4\), which is at (4, 4). Reflect this point over the x-axis to (4, -4). Add seven to the y-coordinate, which brings us to (4, 3). Subtract five from the x-coordinate, which brings us to the treasure at (-1, 3). The treasure at (-1, 3) is exactly four (4) units straight up from the starting point.

Problem 8. If the circle is centered at the origin and tangent to the line \(y = 7\), then its radius is 7 units. The area of the circle is \(\pi \times 7^2 = 49\pi\) square units.
**Problem 9.** This problem could be solved with the Shoestring Algorithm explained in the Club Resource Guide for #10. However, one could also draw the triangle and the rectangle that “circumscribes” it. We can calculate that the area of the rectangular region is $4 \times 3 = 12$ square units. The areas of regions 1, 2, and 3 are 2, 2, and 3 square units, respectively. Therefore, the area of the original (shaded) triangle is $12 - 2 - 2 - 3 = 5$ square units.


**Problem 11.** Due to the sides of the rectangle that are along the lines $x = 1.2$ and $x = 2.6$, we know the rectangle will be $2.6 - 1.2 = 1.4$ units wide (the horizontal sides of the rectangle each will be 1.4 units long). In order to get an area of 14 square units, the length of the rectangle must be 10 units. This means the side opposite the side along $y = -0.2$ must be 10 units above or below $y = -0.2$. In order to get the greatest possible value of $d$, we will go with $y = -0.2 + 10 = 9.8$.

**Problem 12.** The missing vertex of the rectangle must be $(5, -2)$. Notice that there are two each of each $x$-coordinate and each $y$-coordinate among the four vertices. The equation $(x - 5)^2 + (y + 2)^2 = 9$ is the equation of a circle with radius 3 and center at the same point $(5, -2)$. The intersection of the circle and the rectangle is thus a quarter of a circle with radius 3 units. The area of this quarter circle is $(9/4)\pi$ or $(9\pi)/4$ square units.

**Problem 13.** First we will find the distance between the centers of the two circles. The difference in their $x$-values is $17 - 2 = 15$, and the distance between their $y$-values is $10 - 2 = 8$. These lengths are the legs of a right triangle whose hypotenuse can be found using the Pythagorean Theorem: $\sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$ units. Since both circles are tangent to the $x$-axis, the radius of each is equal to its $y$-value. The smaller circle has a radius of 2 units, and the larger circle has a radius of 10 units. The distance between the closest points of the two circles is the distance between their centers minus their radii. That's $17 - 2 - 10 = 5$ units.

**Problem 14.** Since we are rotating 90 degrees, we know that the angle formed with the start point $(4, 1)$, center point $(1, 0)$ and end point will be a right angle. If the angle is 90 degrees, then the two sides of the angle are perpendicular and have slopes that are opposite reciprocals of each other. Notice we travel up 1 unit and right 3 units to go from the center $(1, 0)$ to the point $(4, 1)$. This is a slope of $1/3$. To create a segment perpendicular and of equal length to this one we must move down 3 units and right 1 unit or up 3 units and left 1 unit from the vertex $(1, 0)$. Notice both of these movements create a slope of -3/1 and a segment of equal length to the one created by going from $(1, 0)$ to $(4, 1)$. Going down 3 units and right 1 unit does not get us going in the correct direction, so let's try going up 3 units and left 1 unit from the center $(1, 0)$. This puts us at the point $(0, 3)$ which is in the correct direction and the correct distance from $(1, 0)$.