Topic
The primary focus for this particular meeting is on counting methods to use when there are restrictions on what is being counted. Using the problem-solving techniques of “solve an easier problem,” “look for patterns” and “find the opposite of what is being asked for,” students will solve problems similar to “How many positive three-digit integers have at least one zero?”

Materials Needed
♦ Computer capable of showing the MATHCOUNTS Mini - December 2010 video from www.mathcounts.org (8 min 34 sec)
♦ Copies of the MATHCOUNTS Mini - December 2010 Activity (downloaded from www.mathcounts.org)
♦ Copy of the MATHCOUNTS Mini - December 2010 Activity Solutions (downloaded from www.mathcounts.org)
♦ Calculators

Meeting Plan
This meeting plan makes use of the incredible MATHCOUNTS Mini videos available for free on www.mathcounts.org. Richard Rusczyk (from Art of Problem Solving) hosts each MATHCOUNTS Mini, in which he creatively teaches a concept from a past MATHCOUNTS competition, handbook or poster problem. For this particular meeting plan, we have selected the December 2010 Mini. However, each Mini comes with its own Activity Sheet and Activity Sheet solutions, and you can certainly browse through all of them to see if there are others your students might also enjoy.

From the MATHCOUNTS home page (www.mathcounts.org), select the Videos option from the Features box at the upper right. The MATHCOUNTS Minis (and their accompanying Activity Sheets and Solutions) are then available in the Video Archive in the 2010-2011 Minis section.

For this club activity, students should start off by looking at the MATHCOUNTS Mini - December 2010 Activity Sheet (a small version appears on the next page). The five questions in the Warm-Up! portion have students solve some problems allowing them to use counting methods with which they are currently comfortable. Also in this section, a problem introduces the concept of a subset (which is important for the problem explained in the video).

Now have students watch the MATHCOUNTS Mini - December 2010 video. Rusczyk will explain (in a very entertaining way) different problem solving-techniques he uses (solve an easier problem, look for patterns and find the opposite of what is being asked for) when solving the question in The Problem section.

How many subsets of the set \{M, A, T, H, C, O, U, R, S, E\} contain at least one vowel?
Once the video is complete, students can work together on the Follow-Up Problems, which involve the same concepts discussed in the video. For very advanced students who are familiar with binomial coefficients, the Further Exploration may be of interest.

A glimpse of the first page of the MATHCOUNTS Mini - December Activity Sheet is shown here, as well as the first page of the Activity Sheet Solutions. Both documents can be found in the MCP Members Only section of www.mathcounts.org or as links below the Mini video.
Warm-Up!

Try these problems before watching the lesson.

1. Sam flips a coin 7 times. How many different 7-flip sequences are possible?

2. A set is a collection of different objects, such as numbers or people. A subset of a set is a collection of different objects that are in the original set. So, subsets of the set \{1, 2, 3, 4, 5\} include \{1, 2\} and \{1, 4, 5\}. We also say that the “empty set,” which has no elements at all, is a subset of every set. How many subsets does the set \{Michelle, Laura, Hillary, Barbara, Nancy\} have?

3. (a) How many three-digit numbers are multiples of 7?
   (b) How many three-digit numbers are not multiples of 7?

4. How many five-digit numbers have at least one zero?

5. How many numbers between 150 and 350 are not perfect squares?

The Problem

How many subsets of the set \{M, A, T, H, C, O, U, R, S, E\} contain at least one vowel?
Follow-up Problems


7. How many three-digit numbers have exactly one zero?

8. How many five-letter “words” with at least one vowel can be constructed from the letters A, C, E, G? (Letters may be used more than once in the word.)

9. How many four-digit numbers are there such that the thousands digit is double the units digit?

10. There are 9 children in Mrs. T’s English class, 5 girls and 4 boys. In how many ways can Mrs. T seat them in a row of 9 chairs such that at least 2 girls are next to each other?

11. How many sequences of five letters are such that no two adjacent letters are the same?

12. What percentage of four-digit numbers have a repeated digit? Express your answer as a decimal to the nearest tenth.

Further Exploration

The following problem is appropriate for students who are familiar with binomial coefficients.

13. Explain how to use counting the subsets of a set with \(n\) items to prove that

\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

Wow! Share Your Thoughts

Have some thoughts about the video? Want to discuss the problems on the Activity Sheet? Visit the MATHCOUNTS Facebook page or the Art of Problem Solving Online Community (www.artofproblemsolving.com).
Warm-Up!
1. There are two possible outcomes for the first flip (H, T). For each of these, there are two possible outcomes for the second flip; this is a total of four sequences (HH, HT, TH, TT). For each of these, there are two possible outcomes for the third flip; this is a total of eight sequences. Repeating this multiplication by 2, we can see that the total number of possible 7-flip sequences is \(2^7 = 128\).

2. Let’s rewrite the set as \{M, L, H, B, N\}. The empty set is 1 subset. There are 5 subsets of one element (or person) each: \{M\}, \{L\}, \{H\}, \{B\}, \{N\}. How many two-element subsets are there? Or, how many groups of two can be selected from a group of 5? The answer can be represented as “five choose two” or \(\binom{5}{2} = \frac{5!}{(2!)(3!)} = 10\). Similarly, how many three-element subsets are there? This will also be 10 since for every two-element subset we included, there was a complementary three-element subset we left out. (In other words, when we counted \{M, L\} as a two-element subset, we simultaneously found \{H, B, N\} as a three-element subset.) Similarly, there are 5 subsets of four elements (to complement the 5 subsets of one element) and 1 subset of five elements (to complement the 1 empty set). The total is \(1 + 5 + 10 + 10 + 5 + 1 = 32\) subsets.

3a. Dividing 100 (the smallest possible positive three-digit integer) by 7, we see the answer is a little more than 14. Therefore \(15 \times 7\) is the smallest positive three-digit multiple of 7. Dividing 999 (the largest possible positive three-digit integer) by 7, we see the answer is a little more than 142, so the largest positive three-digit multiple of 7 is \(142 \times 7\). Using 15 through 142, there are \(142 - 14 = 128\) positive three-digit multiples of 7.

3b. There are 999 – 99 = 900 positive three-digit integers. Therefore, 900 – 128 = 772 of them are not multiples of 7.

4. The positive five-digit integers are 10,000 through 99,999. There are 99,999 – 9999 = 90,000 of them. Let’s count how many of them do not have any zeros. The first digit could be any digit from 1 through 9. Similarly, each of the four digits after the first is limited to those nine options since we don’t want any zeros in our number. This means there are \(9 \times 9 \times 9 \times 9 \times 9 = 9^5 = 59,049\) positive five-digit integers that do not have any zeros. Therefore, there are 90,000 – 59,049 = \(30,951\) positive five-digit integers that have at least one zero.

5. Rather than finding the numbers that are not perfect squares, let’s figure out which numbers are perfect squares within the range. The perfect square \(12^2 = 144\) is just too small. The first perfect square within the range is \(13^2 = 169\). Similarly, \(19^2 = 361\) is just too big, but \(18^2 = 324\) is within the range. Therefore, the perfect squares \(13^2\) through \(18^2\) are within the range. That’s 18 – 12 = 6 perfect squares. The range only includes the numbers between 150 and 350, so that’s 349 – 150 = 199 numbers. We’ve determined 6 of them are perfect squares, so 199 – 6 = \(193\) are not perfect squares.

The Problem is solved in the MATHCOUNTS Mini.

Follow-up Problems
6. A subset of the set \{M, A, T, H, C, O, U, R, S, E\} may contain 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 or 10 letters. As with the problem presented in the MATHCOUNTS Mini, consider that each of the 10 letters must choose whether to be in the subset or not. That means there are a total of
2^{10} = 1024 subsets. To determine the number of subsets with at least two letters we need to exclude the empty set and those subsets containing only one letter. That includes \(_{10}C_0 + _{10}C_1 = 1 + 10 = 11\) subsets. Therefore, the number of subsets that contain at least two letters is \(1024 - 11 = 1013\) subsets.

7. There are a total of 999 – 99 = 900 positive three-digit numbers. If exactly one digit must be zero it can only be the tens digit or the units digit. So we can select which digit is zero in 2 ways. The remaining two digits cannot be zero, therefore, they can be chosen in \(9 \times 9\) different ways. The total number of positive three-digit numbers that contain exactly one zero is then \(2 \times 9 \times 9 = 162\).

8. With no restrictions, each letter in the five-letter “word” can be one of four letters, so there are \(4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024\) possible five-letter words. Let’s see how many of these words have no vowels. There would be only two choices for each letter (C or G), so there are \(2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32\) possible five-letter words. This means there are \(1024 - 32 = 992\) five-letter words with at least one vowel.

9. Let X represent the units digit which makes the thousands digit 2X. Since \(0 \times 2 = 0\) and the first digit cannot be zero we can exclude 0 from the possible values of X. If X = 5 we have \(2(5) = 10\), a two-digit number and thus not a candidate for the first digit. For this reason X cannot have a value greater than 4. Thus the possible values of X that would result in a positive four-digit number are 1, 2, 3 and 4. In each of these four cases the tens digit and hundreds digit can be any of the ten numerals 0 to 9, and the thousands digit is pre-determined by the units digit. It follows that there are \(1 \times 10 \times 10 \times 4 = 400\) positive four-digit numbers such that the thousands digit is double the units digit.

10. With 9 students there are a total of 9! = 362,880 different ways to seat them in a row. There are many seating arrangements with at least 2 girls next to each other. Consider, instead, the number of arrangements where no girls are seated next to each other. That only can occur if the seats are arranged G-B-G-B-G-B-G-B-G where G is a girl and B is a boy. Since there are a total of 5 girls in the class they can be seated in 5! different orders. The four boys in the class can be seated in 4! different orders. Thus there are (5!)(4!) = 120 \times 24 = 2,880 different possible seating arrangements in which two girls are not seated next to each other. If we exclude these arrangements from the total number of seating arrangements, we see that there are 362,880 – 2880 = 360,000 ways the students can be seated such that at least two girls are seated next to each other.

11. There are twenty-six letters in the alphabet so the first letter in the sequence can be chosen in 26 different ways. If no two adjacent letters can be the same that leaves only 25 different choices for the second letter in the sequence. Since the third letter cannot be the same as the second letter but may be the same as the first letter, there are again 25 choices for this letter. The same is true for the fourth and fifth letters in the sequence. So there are \(26 \times 25 \times 25 \times 25 \times 25 = 10,156,250\) sequences of five letters in which no two adjacent letters are the same.

12. There are 9,999 – 999 = 9,000 positive four-digit numbers. Rather than counting the amount of four-digit numbers with a repeated digit, we’ll count the amount of four-digit numbers with four distinct digits. There are only 9 ways to select the first digit since it cannot be zero. But the second digit cannot be the same as the first which leaves only 9 ways to select that digit (since it may be zero). If follows that there are 8 ways to choose the third digit since it cannot be the
same as the first or second digits. Finally there are 7 ways to choose the fourth digit since it cannot be the same as the first, second or third digits. Therefore, there are a total of $9 \times 9 \times 8 \times 7 = 4,536$ positive four-digit numbers that have no digit repeated. That means there are $9,000 - 4,536 = 4,464$ positive four-digit numbers in which a digit is repeated. That represents $4,464/9,000 = 0.496 = 49.6\%$ of all positive four-digit numbers.

### Further Exploration

13. Suppose a set has $n$ items. Recall from the MATHCOUNTS Mini presentation the table used to count the number of subsets of various sized sets.

<table>
<thead>
<tr>
<th># terms in subset</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>{M}</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>{M, A}</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>{M, A, T}</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>{M, A, T, H}</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>16</td>
</tr>
</tbody>
</table>

Given a set of $n$ items, to determine the number of subsets containing one term we essentially are calculating $n \choose 1$. Similarly, to determine the number of subsets with two, three, ..., $n$ items we calculate $n \choose 2$, $n \choose 3$, ..., $n \choose n$. For all subsets there is only one subset with zero terms, the empty set which corresponds to $n \choose 0 = 1$. When we sum the number of items contained in all the subsets of a set of $n$ items we see that the total is equivalent to $2^n$.

As presented in the Mini, there is another way to think of the creation of subsets of a set of $n$ items. Let $S$ be a set of $n$ items. When creating a subset of $S$ each term in the set, $s_1, s_2, ..., s_n$ has two choices, to be in the subset, or not to be in the subset.

$$2 \times 2 \times 2 \times 2 \times \ldots \times 2 = 2^n$$