Hat Puzzles Meeting
(Symmetry)

Topic
This meeting’s topic is symmetry across diagonal, horizontal and vertical lines of symmetry.

Materials Needed
- MATHCOUNTS hat poster (included in the Club in a Box resource kit)
- Copies of the two-page hat puzzle (included in the Club in a Box resource kit), with the pieces cut out
- Copies of the horizontal-line-of-symmetry and vertical-line-of-symmetry hat puzzles (available for download from www.mathcounts.org/MCP0708Resources), with the pieces cut out
- Blank puzzle boards - 2 per student/group (available for download from www.mathcounts.org/MCP0708Resources) - Optional
- Scissors - Optional

Helpful Hints
The hat puzzle provided in your Club in a Box resource kit (shown here) matches the poster (shown above). The first page of the puzzle is the “game board,” and the second page has the puzzle pieces that must be cut out. Please feel free to make as many color copies of the puzzle as you like. If you make many copies, put 1s on the back of one game board and its pieces, put 2s on the back of the second game board and its pieces, etc. This will be very important if you have multiple games floating around. Additionally, we recommend laminating the puzzle and its pieces before using this activity with your students but after you have number-coded them on the back. Once a game set (a board and its pieces) has been labeled and/or laminated, it will fit nicely inside a 1-gallon resealable plastic bag.

**This puzzle is challenging. We strongly recommend starting with two easier puzzles we have available online (shown below). These puzzles have vertical and horizontal lines of symmetry and are much easier for students. You can download these puzzles at www.mathcounts.org/MCP0708Resources. We recommend putting a red dot on the back of your pieces for the horizontal-line-of-symmetry puzzle, a yellow dot on the back of your pieces for the vertical-line-of-symmetry puzzle and no dot on the back of your diagonal-line-of-symmetry puzzle (before laminating them). Again, coding the pieces to their corresponding board is also helpful. This way you can easily figure out which pieces go with which board once your students start unintentionally getting all of the pieces mixed together.

Visit www.mathcounts.org/club for the solution and extensions to this problem.
**Meeting Plan**

Before handing out the puzzles, discuss with students what it means to have symmetry. Showing them the puzzle boards, ask them which squares we already know how to fill given the hats that have already been provided. For the diagonal-line-of-symmetry puzzle, what does it mean when the line goes through a square? (*The hat in that square does not have a match. It would match with itself.*)

Be sure students understand we are trying to match only colors across the line of symmetry. The orientation of the hats will not be changing, even though if this were a real mirror image, the orientation of the hats would change. We are concerned only about placing the correct colors in the squares, and all of the hats must remain upright. In the diagonal-line-of-symmetry puzzle, this means that since there is a yellow hat in the upper right corner, we must place a yellow hat in the bottom left corner. See the examples to the right. Though the colors are correct in the first example, the orientation of the hats must always be upright for this game, as they are in the second example.

Students can work in groups on the three puzzles. Working on one puzzle at a time is preferable. Once most of the groups have the answer, ask one group to share its solution. Which piece did they place first and why? (Remember: We recommend doing the horizontal-line-of-symmetry and vertical-line-of-symmetry puzzles before the diagonal-line-of-symmetry puzzle.) Solutions to all three puzzles can be found at www.mathcounts.org/MCP0708Resources.

**Possible Next Steps**

Ask students to create their own puzzles for the next meeting. Or, if you have a relationship with another school in your area with a MATHCOUNTS Club Program, your clubs can swap puzzles that they have created. You can set as many or as few parameters on them as you want. You can let students choose the size of their puzzle board, the orientation of their line of symmetry, the items they are placing in the boards, whether it should be created on the computer, etc. You also can provide your students with two blank boards (available for download at www.mathcounts.org/MCP0708Resources). One of the blank boards can be used to create the board from which the puzzle starts, and the other blank board can be the board they cut up to make their pieces. At the next meeting you can have challenges to see who created the most difficult puzzle to solve. You also can laminate their final products and use them with your classes when you discuss symmetry or use them with your club next year! If students created their puzzles on the computer, ask them for a copy of the file for your records. Students can put their name on the game board so they continue to get credit for their creativity for years to come!

If your students would like to share their puzzles with us, please have them e-mail an electronic version to info@mathcounts.org with the subject line “MATHCOUNTS Club Program.” Thank you!
What are the colors of the caps in the bottom row?

The poster uses caps that can’t be rotated. Therefore, we may not rotate the pieces here with the circles. Our line of symmetry is the key to this problem. We know that any cap can go into a square that the line of symmetry goes through because it must only “match” with itself. However, because of the line of symmetry, we know that a yellow cap must be in the bottom left corner, a blue cap must be right above it and a red cap must be in the first square of the second row. Of the pieces that we are given to use, only two of them have yellow caps. One of them must be positioned so that a yellow cap is in the bottom left corner. Let’s take the piece with three vertical hats (yellow at bottom) and place it in the first column leaving the top square of the column empty. This actually perfectly matches the symmetry of the three original hats that were already placed. We now have the matching red, blue and yellow hats in place. The top left corner square is open now, and we can see that only the piece with the single green hat fits in the spot (Figure 2). Here’s where this falls apart, though. Notice that there are only two blue hats in our remaining pieces. Either they are both in squares along the line of symmetry, or neither of them are on the line of symmetry and they are opposite/symmetric to each other. The only two possibilities are shown in Figure 3. Notice that there are not possible fits for the remaining two pieces in either of these cases. This tells us that our initial placing of that piece with the yellow hat was the wrong choice. It forced us into this dead end.

We need to place the other piece with the yellow hat so that the yellow hat is in the bottom left corner (Figure 4). Now there is only one piece that allows us to have symmetry with the new red hat we just added.

So we place the piece with three (top red) into the fourth column (Figure 5). Notice that we have two pieces left with blue hats, and two mandatory spots in the figure that must have blue hats to complete the symmetry. If we place the piece with two hats (green and blue) in the first column to match the blue hat in the first row, notice that the newly placed green hat doesn’t match properly with the original red hat. This tells us that this particular two-hat piece must be placed so that the blue hat is in the bottom row. The green hat is on the line of symmetry and does not have to match up with any other hat (Figure 6). Now the remaining piece with a blue hat fits nicely into the first column (and its red and green hats match up nicely, too). This leaves obvious placements for the final two pieces, and we maintain our symmetry (Figure 7). This poster is easiest to do if students can actually cut out the pieces and physically move them around to fit into the cubby holes! The bottom row of hats is yellow, red, blue and yellow.

The hat poster problem was originally provided to schools with the 2004-2005 MATHCOUNTS School Handbook.
Horizontal Line of Symmetry - Hat Puzzle

Solution

We are starting with a board that has three hats placed and a horizontal line of symmetry. Just paying attention to the colors, we need to figure out how to place the accompanying pieces such that there is a horizontal line of symmetry on the board. The orientation of the hats (circles in our solution) must not change; all hats will remain upright (so we may not rotate or flip the pieces we are working with). We are only concerned about color-symmetry.

One of our pieces has two green hats, and since they are the only green hats, they must be each other’s match. The only way this can happen is putting the piece in the first or last column. However, by putting it in the last column, the blue hat would be paired up with the yellow hat. Therefore, the piece with two green hats goes in the first column (Figure 1); the two green hats being each other’s match.

The blue hat in the bottom left corner now needs a match in the upper left corner. The only piece that will work is the piece with the blue and yellow hats (Figure 2).

With this piece now placed, the only way to match the new yellow hat in the second column is with the vertical piece having a red hat and yellow hat (Figure 3).

After this piece is placed, we are limited to one position for the remaining piece with the blue hat (Figure 4) and then one option for the remaining piece with the yellow hat and finally the last red hat can be placed (Figure 5).
Vertical Line of Symmetry - Hat Puzzle

Solution

We are starting with a board that has three hats placed and a vertical line of symmetry. Just paying attention to the colors, we need to figure out how to place the accompanying pieces such that there is a vertical line of symmetry on the board. The orientation of the hats (circles in our solution) must not change; all hats will remain upright (so we may not rotate or flip the pieces we are working with). We are only concerned about color-symmetry.

In order to match the yellow hat in the upper right corner, only the piece with the yellow and red hats will work. We have placed that piece in Figure 1.

With two red hats in place and only two red hats in our remaining pieces, we can see how those two pieces must be placed (Figure 2).

The only remaining piece that can be positioned in such a way to match the green hat that has already been placed is the piece with the blue hat and green hat (Figure 3).

Our remaining two pieces can now be placed in the two openings (Figure 4).