MATHCOUNTS® Problem of the Week Archive

Best of 2019! – December 30, 2019

Problems & Solutions

While there were many problems written and solved for MATHCOUNTS in 2019, below is a selection of some of the best of the year!

2018-2019 School Handbook, #27

A segment with endpoints G(-2, 3) and H(4, 7) is dilated by a scale factor of 2/3 with center of dilation (0, 0). What is the sum of all the coordinates of G' and H'?

When a point is dilated by a scale factor of k about the origin, we multiply both the x- and y-coordinate by k to determine the coordinates of the image. So, the image of G(-2, 3) dilated by a factor of 2/3 about (0, 0) is G'((-2)(2/3), (3)(2/3)) = (-4/3, 2). The image of H(4, 7) dilated by a factor of 2/3 about (0, 0) is H'((4)(2/3), (7)(2/3)) = (8/3, 14/3). The sum of the coordinates of G' and H' is -4/3 + 2 + 8/3 + 14/3 = (-4 + 6 + 8 + 14)/3 = 24/3 = ***8****. Alternatively, we could multiply the sum of the coordinates of G and H by 2/3 to get (2/3)(-2 + 3 + 4 + 7) = (2/3)(12) =* ***8****.*

2018-2019 School Handbook, #76

Esther purchased two rock tumblers and one spy pen for $74. Eli purchased two puzzles and a spy pen for $50. Sabine purchased a rock tumbler and two puzzles for $57. Based on this, how much does one puzzle cost?

If we use r for the price of a rock tumbler, s for the price of a spy pen and p for the price of a puzzle, then we can write a system of three equations: 2r + s = 74, 2p + s = 50, r + 2p = 57. Subtracting the second equation from the third, we get r – s = 7 → r = s + 7. Substituting this into the first equation, we get 2(s + 7) + s = 74, which we can solve for s as follows: 2s + 14 + s = 74 → 3s = 60 → s = 20. Now that we know the spy pen costs $20, we can substitute this into the second equation and solve for p as follows: 2p + 20 = 50 → 2p = 30 → p = 15. One puzzle costs ***$15*** *or* ***15.00****.*

2019 Chapter Sprint Round, #19

Let $p \ominus q = \sqrt{p^2 - q^2}$, and let $p \oplus q = \sqrt{p^2 + q^2}$. What is the value of $(3 \oplus 4) \oplus (20 \ominus 16)$?

$(3 \oplus 4) \oplus (20 \ominus 16) = \sqrt{3^2 + 4^2} \oplus \sqrt{20^2 - 16^2} = 5 \oplus 12 = \sqrt{5^2 + 12^2} =$ ***13****.*

The evaluations of the radicals can be done in any of several ways:

Just crunch numbers: $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$; $\sqrt{20^2 - 16^2} = \sqrt{400 - 256} = \sqrt{144} = 12$; $\sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} =$ ***13****.*

Recognize each square root except the last is re-squared, so $\sqrt{3^2 + 4^2 + 20^2 - 16^2}$ = $\sqrt{9 + 16 + 400 - 256}$ = $\sqrt{169}$ = ***13***.

*Note Pythagorean Triples: 3-4-5 scaled by 1 and 4 to 3-4-5 and 12-16-20, respectively; and lastly 5-12-**13**. The ⊕ operation combines two legs to yield the hypotenuse. The ⊖ operation combines the hypotenuse and one leg to yield the other leg.*

<u>2019 State Sprint Round, #24</u>

Eliza creates a custom 6-sided die by randomly choosing six distinct integers from 1 to 7, inclusive, to paint onto the sides of a blank cube. She tells Philip that the faces of her die have a sum of 24. Philip rolls the die. What is the probability that Philip's die shows a prime number on the top face? Express your answer as a common fraction.

*We have for the six faces the sum of the integers 1 through 7, inclusive, except for the one missing value m, and that sum is given to be 24. Therefore, 24 = 1 + 2 + 3 + 4 + 5 + 6 + 7 – m = 28 – m. So, m = 4, and the six faces are 1, 2, 3, 5, 6, 7, of which four values (2, 3, 5, 7) are prime, making the probability of rolling a prime to be **2/3**.*

<u>2019 State Target Round, #8</u>

The sequence a_n is defined by $a_1 = 20$, $a_2 = 19$ and for $n \geq 3$, $a_n = |a_{n-1}| - |a_{n-2}|$. What is the value of a_{2019}?

We are told that $a_1 = 20$ and $a_2 = 19$. We can derive the remainder of the terms of the sequence using the definition $a_n = |a_{n-1}| - |a_{n-2}|$. The list below shows the first 12 terms of the sequence:

$a_1 = 20$ $a_2 = 19$ $a_3 = -1$

$a_4 = -18$ $a_5 = 17$ $a_6 = -1$

$a_7 = -16$ $a_8 = 15$ $a_9 = -1$

$a_{10} = -14$ $a_{11} = 13$ $a_{12} = -1$

We are taught to look for patterns, but we must be careful about assuming the "most obvious" pattern is correct, especially when dealing with non-linear behavior – absolute value might look to be close to linear, but it most certainly is not. The repeated increase by 2 in the left column (except for minor sign issue of the first term), decrease by 2 in the middle column and apparent constancy of the right column might not hold when the left or middle column crosses 0. Let's use this pattern to accelerate getting near this crossing:

$a_{28} = |-1| - |3| = 1 - 3 = -2$ $a_{29} = |-2| - |-1| = 2 - 1 = 1$ $a_{30} = |1| - |-2| = 1 - 2 = -1$

$a_{31} = |-1| - |1| = 1 - 1 = 0$ $a_{32} = |0| - |-1| = 0 - 1 = -1$ $a_{33} = |-1| - |0| = 1 - 0 = 1$

Notice the sudden sign flip. Let's continue to see what happens.

$a_{34} = |1| - |-1| = 1 - 1 = 0$ $a_{35} = |0| - |1| = 0 - 1 = -1$ $a_{36} = |-1| - |0| = 1 - 0 = 1$

*We have two consecutive rows with the exact same sequence of values, and this pattern will continue to hold. Because 2019 is divisible by 3, a_{2019} falls in the right column, with value **1**.*

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