MATHCOUNTS® Problem of the Week Archive

Best of 2020 School Competition – December 7, 2020

Problems & Solutions

While there is no traditional School Competition in this year's Competition Series, this year's online Practice Competitions are in full swing. For some additional practice, see some of last year's School Competition questions below!

Sprint Round, 19

Kendra surveyed 100 people who own a dog or a cat, or both. Of those surveyed, 15 own both a dog and a cat, and the number of people who own a dog is four times the number of people who own a cat. How many people surveyed own a cat?

We can use an equation to represent the provided scenario, $(d + 15) = 4(c + 15)$, where d = the number of people who own a dog and c = the number of people who own a cat. We can simplify this equation as follows: $d + 15 = 4c + 60 \rightarrow d = 4c + 45$. We also know that Kendra surveyed a total of 100 people, so $c + d + 15 = 100$. If we substitute $4c + 45$ in for d, we get: $c + (4c + 45) + 15 = 100 \rightarrow 5c + 60 = 100 \rightarrow 5c = 40 \rightarrow c = 8$. So, if we take the 8 people who own only a cat and the 15 people who own both a cat and a dog, we get $8 + 15 =$ ***23*** *people who own a cat.*

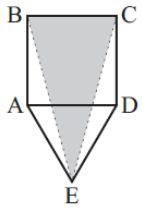
Target Round, 5

A standard, six-sided die is rolled five times. What is the probability that the five rolls are either all the same or all different? Express your answer to the nearest hundredth of a percent.

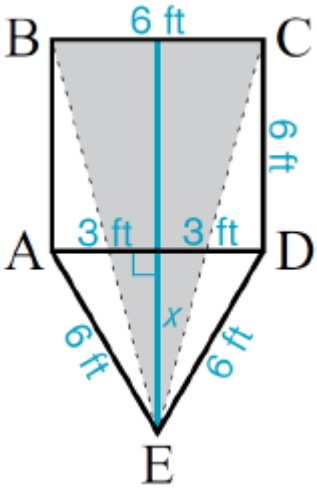
Rolling a six-sided die five times produces $6^5 = 7776$ possible combinations of rolls, since there are 6 possible results in each of the five rolls. We're being asked to look specifically at the situations where all of the rolls are the same and where all of the rolls are different. We can deduce that there are exactly 6 scenarios where all of the rolls are the same: 11111, 22222, 33333, 44444, 55555 and 66666. To represent the scenarios where all of the rolls are different, we can use 6!. This shows that on the first roll, the result could be any of the 6 numbers on the die. However, since all of the rolls must produce different results, there are now only 5 possibilities for the second roll in order to satisfy this requirement. By the same logic, there are only 4 possibilities for the third roll, and so on. To simplify, $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$ possibilities. Since there are only 5 rolls in this scenario, we do not multiply by 1 at the end of this factorial expansion. However, we achieve the same result either way in this case, so we do not need to modify our 6! result in any way. Therefore, there are $720 + 6 = 726$ possible combinations of rolls where either all are the same or all are different. So, the probability of this happening is 726/7776, which is approximately equal to ***9.34****%.*

Team Round, 8

In the figure, square ABCD has side length 6 feet, and E is a point in the exterior of the square such that triangle ADE is equilateral. How many square feet are in the area of shaded triangle BEC? Express your answer as a decimal to the nearest tenth.



The formula for the area of a triangle is A = (1/2)bh, where b = base and h = height. So, in order to find the area of the shaded triangle, BEC, we'll need to know the measurements for the base and the height of this triangle. Since the base of BEC is shared with the square ABCD, and we know the side lengths of the square are all 6 feet, we know that the base of triangle BEC is 6 feet long. Because we know that ADE is an equilateral triangle, we can say that a line segment from E to the midpoint of segment AD would be perpendicular to AD and would represent the height of triangle ADE. We also know that since AD is also a side of the square, AD is 6 feet in length, as are AE and DE, since ADE is an equilateral triangle. This would also mean that the two smaller triangles formed by drawing the line segment from E to the midpoint of AD would create two halves of AD, each equal to 3 feet in length, which would serve as the bases of the smaller triangles formed. Using this information and the Pythagorean Theorem, we can find the height of triangle ADE. Since $a^2 + b^2 = c^2$*, we have* $3^2 + b^2 = 6^2 \rightarrow 9 + b^2 = 36 \rightarrow b^2 = 27 \rightarrow b = \sqrt{27}$ *feet. If we extend the line segment between E and the midpoint of AD all the way down to BC, we are representing the height of the shaded triangle BEC, since ADE is an equilateral triangle that is exactly aligned with the perfect square ABCD. So, since the square has a height of 6 feet, we can find the height of triangle BEC by adding* $6 + \sqrt{27}$*. Using this information and the formula for the area of a triangle, we can find the area of BEC. We have* $A = (1/2)(6)(6 + \sqrt{27}) \rightarrow A = 3(11.1961524227) \rightarrow A = 33.5884572681$ ft^2*. So, the area of triangle BEC is* ***33.6*** ft^2*, rounded to the nearest tenth.*



Countdown Round, 14

What is the sum of the values of *a* that satisfy the equation: $(3)5^2 - 4(5 - a)^2 \div 3 = 63$?

We can start by simplifying this equation: $(3)5^2 - 4(5 - a)^2 \div 3 = 63 \rightarrow 75 - 4(5 - a)^2 \div 3 = 63$*. Subtracting 75 from both sides of the equation gives* $-4(5 - a)^2 \div 3 = -12$*, and then multiplying by 3 on both sides of the equation gives* $-4(5 - a)^2 = -36$*. Next, dividing by –4 on both sides of the equation gives* $(5 - a)^2 = 9$*, and taking the square root of both sides of the equation gives* $5 - a = \pm 3$*. If* $5 - a = 3$*, then* $a = 2$*. If* $5 - a = -3$*, then* $a = 8$*. Therefore, the sum of the values of a that satisfy this equation is* $2 + 8 =$ ***10****.*

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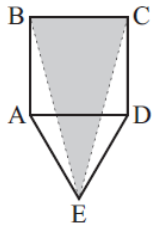

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