MATHCOUNTS® Problem of the Week Archive

Happy Numbers – August 29, 2016

Problems & Solutions

For any positive integer, find the sum of the squares of its digits. If the result is 1, you're done. Otherwise, find the sum of the squares of the digits of the result. Continue this process until the result is 1, in which case we call the original integer a *happy number*, or until there is a repeated result, which leads to an endless loop, in which case we call the original integer an *unhappy number*. For example, consider 13 and 85.

13 is a happy number since
$1^2 + 3^2 = 1 + 9 = 10$,
$1^2 + 0^2 = 1 + 0 = 1$.

85 is an unhappy number since
$8^2 + 5^2 = 64 + 25 = 89$,
$8^2 + 9^2 = 64 + 81 = 145$,
$1^2 + 4^2 + 5^2 = 1 + 16 + 25 = 42$,
$4^2 + 2^2 = 16 + 4 = 20$,
$2^2 + 0^2 = 4$, $4^2 = 16$,
$1^2 + 6^2 = 1 + 36 = 37$,
$3^2 + 7^2 = 9 + 49 = 58$,
$5^2 + 8^2 = 25 + 64 = 89$, which is a repeated result.

What is the least unhappy number?

We know 1 is happy so let's look at 2. Following the process of summing the squares of the digits we get $2^2 = 4$, $4^2 = 16$, $1^2 + 6^2 = 37$, $3^2 + 7^2 = 58$ and $5^2 + 8^2 = 89$. We know from the problem statement that a result of 89 will loop endlessly. The smallest unhappy number is ***2****.*

What is the least prime number that is a happy number?

We know 2 is unhappy so the next prime to try is 3. Following the same process, we get $3^2 = 9$, $9^2 = 81$, $8^2 + 1^2 = 65$ and $6^2 + 5^2 = 61$. We can stop here because 61 has the same two digits as 16 so the sum of their squares will be the same. We saw 16 appear when solving to see if 2 was happy, so we know this will endlessly loop as well. Now let's look at 5: $5^2 = 25$, $2^2 + 5^2 = 29$ and $2^2 + 9^2 = 85$. From the problem statement we know that 85 is an unhappy number, which results in an endless loop. Let's try 7: $7^2 = 49$, $4^2 + 9^2 = 97$, $9^2 + 7^2 = 130$, $1^2 + 3^2 + 0^2 = 10$ and $1^2 + 0^2 = 1$. So the smallest prime number that is also a happy number is ***7****.*

What is the least integer greater than 1 that, when multiplied by any happy number, yields another happy number?

If we multiply any number by 1, the number remains the same. If we multiply any of our happy numbers by 10, the new number will have the same digits, plus a new digit of 0, but since $0^2 = 0$, this will not affect the sum of the squares of the digits. We can multiply by any number that starts with a 1 and is followed by 0's, but the smallest number that when multiplied by any happy number yields another happy number is ***10****.*

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