MATHCOUNTS® Problem of the Week Archive

2018 National Champion – May 21, 2018

Problems & Solutions

Last week the National Competition concluded, and Luke Robitaille from Texas became the first ever two-time MATHCOUNTS National Champion. Let's look at some of the problems he had to solve on the way to the top!

In a card game, Nora draws cards at random, without replacement, from a deck of 21 cards. Twenty of the cards are numbered 1 through 20, and the other card is marked "Joker." Nora keeps all of the cards she draws before she draws the Joker. What is the probability that the cards Nora keeps include exactly four prime-numbered cards? Express your answer as a common fraction.

[Sprint #17]

*The eight prime-numbered cards among the cards in this game are 2, 3, 5, 7, 11, 13, 17 and 19. We are interested in the outcomes that result in four of these eight cards being included in the cards Nora keeps. Determining the number of favorable outcomes and the number of total outcomes with permutations of prime- and composite-numbered cards would be too time consuming. Instead we can simply focus on where the Joker is drawn relative to the eight prime-numbered cards. There are nine possible positions in which the Joker can be drawn (JPPPPPPPP, PJPPPPPPP, PPJPPPPPP, ...). There is only one favorable outcome, when it is drawn between the fourth and fifth prime-numbered cards (PPPPJPPPP). So, the probability of that outcome is **1/9**.*

A *4-up number* is defined as a positive integer that is divisible by neither 2 nor 3 and does not have 2 or 3 as any of its digits. How many numbers from 400 to 600, inclusive, are 4-up numbers?

[Target #7]

Since 600 is divisible by both 2 and 3, it is not a 4-up number. Thus, we need only consider three-digit numbers of the forms 4BC and 5BC, where B and C represent the tens and units digit of each, respectively. Since 4-up numbers are not divisible by 2, the possible values for the units digit of each three-digit number are 1, 5, 7 and 9. We now see that possible 4-up numbers from 400 to 600, inclusive are of the forms 4B1, 4B5, 4B7, 4B9, 5B1, 5B5, 5B7 and 5B9. For a number to be divisible by 3, the sum of its digits must be divisible by three. So, we need to determine the number of possible values of B so that the sums 4 + B + 1, 4 + B + 5, …, 5 + B + 7, 5 + B + 9 are not divisible by 3. The possible values of B for each are listed below:
4B1: 0, 5, 6, 8 and 9 are the 5 possible values of B.
4B5: 1, 4, 5, 7 and 8 are the 5 possible values of B.
4B7: 0, 5, 6, 8 and 9 are the 5 possible values of B.
4B9: 0, 1, 4, 6, 7 and 9 are the 6 possible values of B.
5B1: 1, 4, 5, 7 and 8 are the 5 possible values of B.
5B5: 0, 1, 4, 6, 7 and 9 are the 6 possible values of B.
5B7: 1, 4, 5, 7 and 8 are the 5 possible values of B.
5B9: 0, 5, 6, 8 and 9 are the 5 possible values of B.
*That's a total of 5 + 5 + 5 + 6 + 5 + 6 + 5 + 5 = **42** numbers.*

Captain Greenbeard's treasure chest holds 100 grams of a mix of gold, silver and bronze coins. The gold, silver and bronze coins each weigh 6 grams, 2 grams and 5 grams, respectively, and are worth 30 droubles, 8 droubles and 15 droubles, respectively. Greenbeard has at least one of each kind of coin. What is the greatest possible total value, in droubles, of the coins in the chest?

[Team #3]

*Since we are told that Greenbeard has at least one of each kind of coin, that accounts for 6 + 2 + 5 = 13 grams of the total 100 grams held in the treasure chest. Let's determine the maximum possible value of the remaining 100 – 13 = 87 grams. The gold coins are the most valuable, so we want as many of those as possible. At a weight of 6 grams each, there could be at most another 14 gold coins, for an additional 14 × 6 = 84 grams. But there is no way to reach a total weight of exactly 100 grams with additional silver and bronze coins. If, however, there were another 13 gold coins, for an additional 13 × 6 = 78 grams, that would leave 87 – 78 = 9 grams to account for. Since the bronze coins are more valuable than the silver coins, let's determine the maximum number of additional bronze coins next. The remaining 9 grams of weight could come from at most one additional bronze coin weighing 5 grams. That leaves 9 – 5 = 4 grams to account for. Two silver coins weighing 2 grams each could account for the remaining 4 grams. That's 14 gold coins, 2 bronze coins and 3 silver coins, with a total weight of 14 × 6 + 2 × 5 + 3 × 2 = 84 + 10 + 6 = 100 grams. This yields the largest possible value of the coins in the treasure chest, which is 14 × 30 + 2 × 15 + 3 × 8 = 420 + 24 + 30 = **474** droubles.*

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