

MATHCOUNTS[®] Problem of the Week Archive

2019 National Champion – May 20, 2019

Problems & Solutions

Last week the national competition concluded, and Daniel Mai from Massachusetts earned the title of MATHCOUNTS National Champion. Let's look at some of the problems he had to solve on the way to the top!

Sprint #14

Two opposite vertices of a certain square are located at $(1, 6)$ and $(-3, 1)$. If the line $y = mx$ divides this square into two regions of equal area, what is the value of m ? Express your answer as a common fraction.

Because of symmetry, any line that separates the square into two regions of equal area must intersect the center point of the square. Since the segment with endpoints $(-3, 1)$ and $(1, 6)$ is a diagonal of the square, we know that the center point of the square is at the midpoint of this segment, which is $((-3 + 1)/2, (6 + 1)/2) = (-1, 7/2)$. The line given by $y = mx$ intersects the point $(0, 0)$. So, the line in question passes through $(0, 0)$ and $(-1, 7/2)$ and has slope $m = (7/2 - 0)/(-1 - 0) = -7/2$.

Sprint #20

A certain digital lock has a keypad with five buttons labeled 1 to 5. To activate the locking mechanism, a secret code is set with these restrictions: the code must contain three parts, each part will consist of either pressing one button or pressing two buttons simultaneously, and no button may be pressed more than once. For example $[3][4][1]$, $[1][2\&4][5]$ and $[1\&2][4][3\&5]$ are three possible codes. Also, note that $[1][2\&4][5]$ and $[1][4\&2][5]$ are indistinguishable. How many distinct secret codes can be set for this lock?

A code for this lock can consist of three parts that are three presses of a single button, or two presses of a single button and one of two buttons pressed simultaneously, or one press of a single button and two of two buttons pressed simultaneously. Let's examine these three cases.

Case 1: *The number of codes of the form $[a][b][c]$ is $5 \times 4 \times 3 = 60$ codes.*

Case 2: *For codes of the form $[a], [b], [c\&d]$, there are $5 \times 4 = 20$ options for $[a]$ and $[b]$, leaving ${}^3C_2 = 3$ options for $[c\&d]$. That gives us $20 \times 3 = 60$ combinations of the three parts, each of which can be arranged in 3 orders, for a total of $60 \times 3 = 180$ codes.*

Case 3: *For codes of the form $[a][b\&c][d\&e]$, there are 5 options for $[a]$, leaving ${}^4C_2 = 6$ options for $[b\&c]$, which then leaves 1 option for $[d\&e]$, for a total of $5 \times 6 \times 1 = 30$ combinations of the three parts, each of which can be arranged in 3 orders, for a total of $30 \times 3 = 90$ codes.*

Therefore, the total number of distinct codes possible is $60 + 180 + 90 = 330$ codes.

Sprint #29

How many of the first 100,000 positive integers have no single-digit prime factors?

Here is what we get counting multiples of 2, 3, 5 and 7. We must be careful, though, as some integers are counted two, three,

| | | | | |
|------------------|------------------|--------------------|-----------------------|--------------------|
| even four times: | 2: 50,000 | 2 × 3: 16,666 | 2 × 3 × 5: 3333 | 2 × 3 × 5 × 7: 476 |
| | 3: 33,333 | 2 × 5: 10,000 | 2 × 3 × 7: 2380 | |
| | 5: 20,000 | 2 × 7: 7142 | 2 × 5 × 7: 1428 | |
| | 7: <u>14,285</u> | 3 × 5: 6666 | 3 × 5 × 7: <u>952</u> | |
| | 117,618 | 3 × 7: 4761 | 8093 | |
| | | 5 × 7: <u>2857</u> | | |
| | | 48,092 | | |

First, we see that the 48,092 multiples of 6, 10, 14, 15, 21 and 35 were counted twice among the 117,618 multiples of 2, 3, 5 and 7. For example, multiples of $2 \times 3 = 6$ were counted with multiples of 2 and with multiples of 3. Since these integers have been double counted, we subtract and get $117,618 - 48,092 = 69,526$ integers.

Next, the 8093 multiples of 30, 42, 70 and 105 were counted three times among the 48,092 multiples of 6, 10, 14, 15, 21 and 35. For example, multiples of $2 \times 3 \times 5 = 30$ were counted with multiples of $2 \times 3 = 6$, with multiples of $2 \times 5 = 10$ and with multiples of $3 \times 5 = 15$. We just removed all three counts of these 8093 integers when we subtracted the 48,092 integers to resolve the double counting. So, we add these 8093 integers to our previous total and get $69,526 + 8093 = 77,619$ integers.

Finally, the 476 multiples of 210 were counted four times among the 8093 multiples of 30, 42, 70 and 105. For example, multiples of $2 \times 3 \times 5 \times 7 = 210$ were counted with multiples of $2 \times 3 \times 5 = 30$, with multiples of $2 \times 3 \times 7 = 42$, with multiples of $2 \times 5 \times 7 = 70$ and with multiples of $3 \times 5 \times 7 = 105$. We resolved three of the counts of these 476 integers when we initially subtracted 48,092 to resolve double counting, but we brought one of the counts back when we just added 8093 to our previous total. Now, we see that, 476 of those 77,619 integers are counted twice and subtracting gives us a total of $77,619 - 476 = 77,143$ of the first 100,000 positive integers that have a single-digit prime factor. Therefore, the total number of integers that do not have a single-digit prime factor is $100,000 - 77,143 = \mathbf{22,857}$ integers.

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