MATHCOUNTS® Problem of the Week Archive

MATHCOUNTS Valentine – February 11, 2019

Problems & Solutions

On some graph paper, graph the following segments:

$y = x$, for $0 \le x \le 2$
$y = 2x - 2$, for $2 \le x \le 3$
$x = 3$, for $4 \le y \le 6$
$y = -x + 9$, for $2 \le x \le 3$
$y = 7$, for $1 \le x \le 2$
$y = x + 6$, for $0 \le x \le 1$

Now reflect each of the segments over the y-axis. What popular shape have you drawn?

The first segment connects the points (0, 0) and (2, 2). The second segment connects the points (2, 2) and (3, 4). The third segment connects the points (3, 4) and (3, 6). The fourth segment connects the points (3, 6) and (2, 7). The fifth segment connects the points (2, 7) and (1, 7). Finally, the sixth segment connects the points (1, 7) and (0, 6). This should appear as half a heart. Once the reflection is done, you should have the shape of a ***heart*** *with the y-axis running down the center.*

What is the area of the region you have enclosed with this Valentine shape?

One way to determine the area of the entire heart is to determine the area of the right side and then double it. If we focus on the right side of the heart, we can divide the region into a series of triangles, trapezoids and/or rectangles. One approach to dividing the right half of the figure follows. Drawing a segment connecting the points (0, 6) and 3, 6) creates an isosceles trapezoid with a top base, bottom base and height of lengths 1 unit, 3 units and 1 unit, respectively. The area of this region is (1/2)(1 + 3)(1) = 2 units^2. Drawing another segment connecting the points (0, 4) and (3, 4) creates a 3 by 2 rectangle. The area of this region is 3 × 2 = 6 units^2. Drawing a segment from (2, 4) to (2, 2) creates a right trapezoid and a right triangle. The right trapezoid has a height of 2 units and bases of lengths 2 units and 4 units. The area of the this region is (1/2)(2 + 4)(2) = 6 units^2. The right triangle has a height of 2 units and a base of 1 unit. The area of this region is (1/2) × 1 × 2 = 1 unit^2. Adding together the areas of these regions, we see the right half of the heart (using the y-axis as a boundary), has an area of 2 + 6 + 6 + 1 = 15 units^2. Therefore, the area of the entire heart is 2 × 15 = ***30*** *units^2.*

If you were to perform a dilation of the complete enclosed region about the point (0, 4) with a scale factor of 2, what would be the area of this new shape?

Dilations cause a shape to grow or shrink (and sometimes flip!). Since the scale factor was +2, that means that the distance between a point on the new figure and (0, 4) will be twice as long as the distance between the corresponding point on the original figure and (0, 4). For instance, the original point of (0, 0), which is 4 units from (0, 4), will now move 4 units further away to the point (0, −4). This means, too, that all of the segments of the original figure will double in size, which will cause the area of the figure to quadruple in size. Therefore, the area of the new shape will be 4 × 30 = ***120*** *units^2.*

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Now reflect each of the segments over the y-axis. What popular shape have you drawn?

What is the area of the region you have enclosed with this Valentine shape?

If you were to perform a dilation of the complete enclosed region about the point (0, 4) with a scale factor of 2, what would be the area of this new shape?