

MATHCOUNTS *MiniS* December 2013 Activity Solutions

Warm-Up!

1. We are asked to determine the value of the sum $1 + 2 + 3 + \dots + 98 + 99$. Adding pairs of these addends, we notice a pattern. For example, pairing the first and last numbers, we have $1 + 99 = 100$. Then pairing the second number with the next to last number, we see that $2 + 98 = 100$. We will be able to do this for a total of 49 pairs of addends, with the addend of 50 left in the middle unpaired. That means the sum of the first 99 positive integers is $49(100) + 50 = 4900 + 50 = \mathbf{4950}$.

2. If we let S represent the sum of the twelve numbers, then the mean of the twelve numbers can be expressed as $S/12 = -3$. Solving for S , we see that the sum of the twelve numbers is $S = \mathbf{-36}$.

3. We are looking for the smallest possible perfect square multiple of 135. The prime factorization of 135 is $3^3 \times 5$. If we multiply this quantity by one more 3 and one more 5, we can create a perfect square. We get $(3^3 \times 5) \times (3 \times 5) = 3^4 \times 5^2 = (3^2 \times 5)^2 = 45^2$. Thus, the smallest possible multiple of 135 that is a perfect square is $45^2 = \mathbf{2025}$.

4. The factors of 105 are 1, 3, 5, 7, 15, 21, 35 and 105. The ordered pairs (1, 105), (105, 1), (3, 35), (35, 3), (5, 21), (21, 5), (7, 15) and (15, 7) are the **8** ordered pairs that satisfy $xy = 105$.

The Problems are solved in the video.

Follow-up Problems

5. Let the 23 consecutive integers be expressed as $n - 11, n - 10, n - 9, \dots, n - 1, n, n + 1, \dots, n + 10, n + 11$. Applying the concepts from the video, it follows that if the mean of the 23 integers is 14, then the middle integer is $n = 14$. Therefore, the smallest of the 23 integers is $n - 11 = 14 - 11 = \mathbf{3}$.

6. Consider the list of all 4-digit numbers with units digit 1 to be 1001, 1011, 1021, 1031, ..., 9971, 9981, 9991. There are 900 such numbers, and notice that the difference between consecutive numbers in this list is 10. Using what we learned from the video, we can conclude that the arithmetic mean of this list of numbers will be the middle value, which we can determine by adding the first and last numbers in the list and dividing by two. So the arithmetic mean of these numbers is $(9991 + 1001)/2 = 10,992/2 = \mathbf{5496}$.

7. The difference between consecutive terms in any list of the smallest n positive odd integers will be 2, so once again, we can apply the techniques used in the video. The sum of the list of n integers is the same as the arithmetic mean repeated n times. Let's consider two such lists. When $n = 4$, we have 1, 3, 5, 7, and when $n = 5$, we have 1, 3, 5, 7, 9. In a list of the smallest n positive odd integers the first number is always 1, and the final number is $2n - 1$. Thus, the arithmetic mean of such a list (also the middle value) is $(1 + 2n - 1)/2 = n$. Since the sum of the list of integers is the same as the arithmetic mean repeated n times, the sum would be $n \times n = n^2$.

8. Let the 9 consecutive positive multiples of 5 be expressed as $5n - 20$, $5n - 15$, $5n - 10$, $5n - 5$, $5n$, $5n + 5$, $5n + 10$, $5n + 15$, $5n + 20$. The sum of the 9 integers is $45n$. We are told that this sum is a perfect cube, so $45n = y^3$. Since $45 = 3^2 \times 5$, we can write $3^2 \times 5 \times n = y^3$. If we multiply $3^2 \times 5$ by another 3 and two more 5s, we can create a perfect cube. We get $(3^2 \times 5) \times (3 \times 5^2) = 3^3 \times 5^3 = (3 \times 5)^3 = 15^3$. Thus, $n = 3 \times 5^2 = 75$, and the smallest of the 9 integers is $5n - 20 = 5(75) - 20 = 375 - 20 = \mathbf{355}$.

9. The sum of the first k positive integers can be expressed as $(1 + k)/2 \times k$. If this must evenly divide $24k$, then the product of $(1 + k)/2 \times k$ and some integer, m , must equal $24k$. We can write $((1 + k)/2 \times k) \times m = 24k \rightarrow (1 + k)/2 \times m = 24 \rightarrow (1 + k) \times m = 48$. Thus, $1 + k$ must be a factor of 48. Let's determine how many factors of 48 can be written as $1 + k$, where k is a positive integer. The 10 factors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48. Since $1 + k = 1$ when $k = 0$, there are only **9** integer values of k , with $k > 0$, for which the sum of the first k integers evenly divides $24k$.