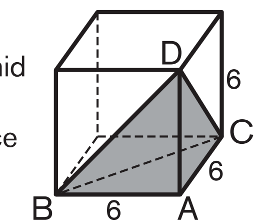
# MATHCOUNTS® Minis March 2020 Activity Solutions

## Warm-Up!

1. A cube with edge length 6 inches has volume $6 \times 6 \times 6 = 216$ in$^3$. A cube with edge length 1 foot = 12 inches has volume $12 \times 12 \times 12 = 1728$ in$^3$. The ratio of these volumes is 216/1728 = **1/8**. We can also determine the ratio of the volumes without actually calculating the volumes. The ratio of the edge lengths of the two cubes is 6/12 = 1/2. Since volume is the product of three linear measures, it follows that the ratio of the volumes will be $1^3/2^3$ = **1/8**.

2. The formula for the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where $h$ and $r$ are the height and the radius of the base of the cone, respectively. We are told that the circumference ($2\pi r$) of the base of the cone is $6\pi$ inches, thus $r = 3$ inches. Since the height of the cone is three times is radius, $h = 3(3) = 9$ inches. We now can substitute to see that the volume of the cone is $\frac{1}{3}\pi(3^2)(9) =$ **27π** in$^3$.

3. Tetrahedron ABCD is shown here. The volume of the tetrahedron (which is a pyramid with a triangular base) is $1/3 \times B \times h$, where $B$ is the area of the base, and $h$ is the height of the tetrahedron. The area of the base ($\Delta$ABC) is $1/2 \times 6 \times 6 = 18$ cm$^2$. Since the height is 6 cm, the volume of the tetrahedron is $1/3 \times 18 \times 6 =$ **36** cm$^3$.




4. The volume of a cylinder is the area of the circular base times the height. If we want the two volumes to be equal, we can set up the following equation $\pi \times (3/2)^2 \times 6 = \pi \times (4/2)^2 \times h$. Solving for $h$, we get $h = 3^2 \times 6 \div 4^2 = 54/16 =$ **3 3/8** inches.

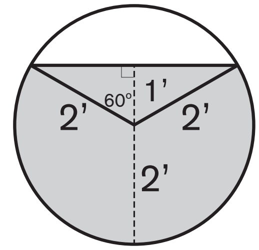
**The Problems** are solved in the MATHCOUNTS® Mini video.

## Follow-up Problems

5. Let $r$ and $h$ represent the radius and height of the cone, respectively. The smallest cone, then, has radius $(1/4)r$ and height $(1/4)h$. The third-largest cone has radius $(1/2)r$ and height $(1/2)h$. The second-largest cone has radius $(3/4)r$ and height $(3/4)h$. The largest of the four pieces is the frustum created by removing the second-largest cone, with volume $(1/3)\pi[(3/4)r]^2(3/4)h = (9/64)\pi r^2 h$, from the largest cone, with volume $(1/3)\pi r^2 h$. So, the largest piece has volume $(1/3)\pi r^2 h - (9/64)\pi r^2 h = [(64 - 27)/192]\pi r^2 h = (37/192)\pi r^2 h$. The second-largest of the four pieces is the frustum created by removing the third-largest cone, with volume $(1/24)\pi r^2 h$, from the second-largest cone, with volume $(9/64)\pi r^2 h$. The second-largest piece, then, has volume of $(9/64)\pi r^2 h - (1/24)\pi r^2 h = [(27 - 8)/192]\pi r^2 h = (19/192)\pi r^2 h$. Therefore, the ratio of the volumes of the second-largest piece and the largest piece is $[(19/192)\pi r^2 h]/[(37/192)\pi r^2 h] =$ **19/37**.

6. The two cross-sections in question are similar hexagons, and their altitudes to the apex of the pyramid (which we can represent with $x$ and $x + 8$) will be in the same ratio as the ratio of their other corresponding linear lengths. The square of the ratio of their corresponding linear lengths is equal to the ratio of their areas, so we can write $[x/(x + 8)]^2 = (216\sqrt{3})/(486\sqrt{3}) = 4/9$. From here we can proceed as follows: $[x/(x + 8)]^2 = 4/9 \rightarrow x/(x + 8) = 2/3 \rightarrow 3x = 2x + 16 \rightarrow x = 16$, and the distance from the apex of the pyramid to the largest cross-section is $x + 8 = 16 + 8 =$ **24** feet.

7. The figure shows the area of the circular base, when the cylinder is on its side, that is covered by the oil. The volume of the oil, will be this area multiplied by the height of the tank when upright. In order to find this area, we need to find the area of the segment of the circle not covered by oil and subtract this from the area of the entire circle. Drawing in our known distances and a few radii, we can see find the angle of the arc of the segment. The perpendicular distance from the center of the circle to the top of the oil is 1 foot, and the radius is 2 feet. We see a right triangle with one side length of 1 and the hypotenuse of length 2. The other leg will be $\sqrt{3}$, these are the side ratios of a 30-60-90 triangle. So the entire angle of the arc is $2 \times 60 = 120$ degrees. The area of the arc is 1/3 of the area of the circle, since 120 degrees is 1/3 of the entire measure of a circle. The area of the segment is the area of the arc minus the area of the triangle or $1/3 \times \pi \times 2^2 - 1/2 \times 2\sqrt{3} \times 1$. This we subtract from the entire area of the circle to get $\pi \times 2^2 - (1/3 \times \pi \times 2^2 - 1/2 \times 2\sqrt{3} \times 1) = 2/3 \times \pi \times 2^2 + 1/2 \times 2\sqrt{3} \times 1 \approx 10.1$ ft$^2$. The volume is $15 \times 10.1 = 151.5$ ft$^3$. We are looking for the height of the oil when the cylinder is upright. We will set up the following equation: $151.5 = \pi \times 2^2 h$. Solving for $h$, we get $h \approx$ **12.1** feet.




8. Starting with cube XBDCAFZE with side length 4 units, shown here, and removing tetrahedron XABC and tetrahedron ZDEF, to create the two equilateral triangular faces ABC and DEF, will leave us with our desired polyhedron. The volume of the cube itself is $4 \times 4 \times 4 = 64$ units$^3$. The volume of the two congruent tetrahedrons are $1/3 \times 1/2 \times 4 \times 4 \times 4 = 10\ 2/3$ units$^3$. So the volume of the polyhedron is $64 - 2 \times 10\ 2/3 = 64 - 21\ 1/3 =$ **42 2/3** units$^3$.