MATHCOUNTS® Minis **February 2020 Activity Solutions**

Warm-Up!

1a. Distributing, we get $(x - 1)(x + 1) = (x - 1)(x) + (x - 1)(1) = x^2 - x + x - 1 = \mathbf{x^2 - 1}$.

1b. Distributing, we get $(x - 2)(x + 2) = (x - 2)(x) + (x - 2)(2) = x^2 - 2x + 2x - 4 = \mathbf{x^2 - 4}$.

1c. Distributing, we get $(x - y)(x + y) = (x - y)(x) + (x - y)(y) = x^2 - xy + xy - y^2 = \mathbf{x^2 - y^2}$.

2a. $5^2 - 4^2 = 25 - 16 = \mathbf{9}$

2b. $6^2 - 5^2 = 36 - 25 = \mathbf{11}$

2c. $7^2 - 6^2 = 49 - 36 = \mathbf{13}$

2d. $8^2 - 7^2 = 64 - 49 = \mathbf{15}$

2e. It appears that the difference of the squares of two consecutive integers is equal to the sum of the two consecutive integers. To see if this is always true, let's do a little algebra. Let $n - 1$ and n be two consecutive integers. What do we get when we simplify $n^2 - (n - 1)^2$? Start by expanding $(n - 1)^2$ to get $n^2 - (n^2 - n - n + 1) = n^2 - n^2 + n + n - 1 = n + n - 1 = n + (n - 1)$, which is the sum of the two consecutive integers. Therefore, **it is the case that the difference of the squares of two consecutive integers equals the sum of the two consecutive integers**.

3. We could solve $29 \times 31 + 19 \times 21$ by multiplying 29×31 and 19×21, then adding the results. But we can save ourselves some time and work by recognizing that $29 = 30 - 1$ and $31 = 30 + 1$ and that 19 = and $21 = 20 + 1$ and recalling from problem 1c above that $(x - y)(x + y) = x^2 - y^2$. We can rewrite the expression as $(30 - 1)(30 + 1) + (20 - 1)(20 + 1) = 30^2 - 1^2 + 20^2 - 1^2 = 900 - 1 + 400 - 1 = 1300 - 2 = \mathbf{1298}$.

4. Let our positive integers be called x and y, with $x > y$. We know that $x + y = 11$ and that $x^2 - y^2 = 55$. We are trying to determine the value of $x - y$. Once again, recall from problem 1c that $x^2 - y^2 = (x - y)(x + y)$. Let's rewrite our second equation as $(x - y)(x + y) = 55$. Substituting 11 for $x + y$, we get $11(x - y) = 55$, so $x - y = \mathbf{5}$.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Notice that our expression is the difference of squares since $2^{1024} - 1 = (2^{512})^2 - 1^2$. We can rewrite $(2^{512})^2 - 1^2$ as $(2^{512} - 1)(2^{512} + 1)$. The second factor is very large, but the first factor is smaller and is also the difference of squares, namely $(2^{256})^2 - 1^2$. This expression can be rewritten as $(2^{256} - 1)(2^{256} + 1)$. Again, the first factor is smaller than the second factor and can be rewritten because it is the difference of squares, namely $(2^{128})^2 - 1^2$. If we continue rewriting the expression in this manner, we eventually get to a factor of $2^4 - 1^2 = (2^2 - 1)(2^2 + 1) = (3)(5)$. Neither of these factors can be broken down further. The only smaller prime that might be factor is 2. However, $2^{1024} - 1$ is odd and doesn't have a factor of 2. So, 3 and 5 are the two smallest prime factors, and their product is $3 \times 5 = \mathbf{15}$.

6. Let our positive integers be x and y, with $x > y$. We know that $x^2 - y^2 = 2009$, so $(x + y)(x - y) = 2009$. To maximize the value of $x - y$, let's try $y = 1$. Substituting, we get $x^2 - 1^2 = 2009$, so $x^2 = 2010$. However, $x = \sqrt{2010}$ is not an integer. The smallest perfect square that is greater than 2009 would be 2025. So, to get $x = \sqrt{2025} = 45$, we must have $x^2 = 2025 = 2009 + 16$. Therefore, $x^2 - 16 = 2009$ and $x^2 - 4^2 = 2009$. In this case, $x = 45$ and $y = 4$ and $x - y = 45 - 4 =$ **41**. (Note: the other x and y that satisfy the equation, $x = 147$ and $y = 140$, have difference $147 - 140 = 7$.)

7. Since a, b and c are consecutive positive odd integers, we can write a and c in terms of b as $a = b - 2$ and $c = b + 2$. We know that $b^2 - (b - 2)^2 = 344$, and $(b + 2)^2 - b^2$ is what we are asked to find. Let's start by simplifying $b^2 - (b - 2)^2 = 344$. We have $b^2 - [(b - 2)(b - 2)] = 344 \rightarrow b^2 - (b^2 - 4b + 4) = 344 \rightarrow b^2 - b^2 + 4b - 4 = 344 \rightarrow 4b - 4 = 344 \rightarrow 4b = 348$. Next, let's simplify $(b + 2)^2 - b^2$. We get $(b + 2)(b + 2) - b^2 \rightarrow b^2 + 4b + 4 - b^2 \rightarrow 4b + 4$. Since $4b = 348$, we have $4b + 4 = 348 + 4 =$ **352**.

8. Let m and p represent Minnie's and Paul's ages, respectively. We know that $m + p^2 = 7308$ and $p + m^2 = 6974$. Subtracting these two equations yields $m + p^2 - p - m^2 = 334$, which can be rewritten as $-(p - m) + (p - m)(p + m) = 334$ or $(p - m)(-1 + p + m) = 334$. Since $334 = 2 \times 167$ and 167 is prime, it follows that $p - m = 2$ and $-1 + p + m = 167$. Therefore, $p + m =$ **168**.