Warm-Up!

1a. Distributing, we get \((x - 1)(x + 1) = x^2 - x + x - 1 = x^2 - 1\).

1b. Distributing, we get \((x - 2)(x + 2) = x^2 - 2x + 2x - 4 = x^2 - 4\).

1c. Distributing, we get \((x - y)(x + y) = x^2 - xy + xy - y^2 = x^2 - y^2\).

2a. \(5^2 - 4^2 = 25 - 16 = 9\)

2b. \(6^2 - 5^2 = 36 - 25 = 11\)

2c. \(7^2 - 6^2 = 49 - 36 = 13\)

2d. \(8^2 - 7^2 = 64 - 49 = 15\)

2e. It appears that the difference of the squares of two consecutive integers is equal to the sum of the two consecutive integers. To see if this is always true, let's do a little algebra. Let \(n - 1\) and \(n\) be two consecutive integers. What do we get when we simplify \(n^2 - (n - 1)^2\)? Start by expanding \((n - 1)^2\) to get \(n^2 - n^2 + n - n - 1 = n + n - 1 = n + (n - 1)\), which is the sum of the two consecutive integers. Therefore, it is the case that the difference of the squares of two consecutive integers equals the sum of the two consecutive integers.

3. We could solve \(29 \times 31 + 19 \times 21\) by multiplying \(29 \times 31\) and \(19 \times 21\), then adding the results. But we can save ourselves some time and work by recognizing that \(29 = 30 - 1\) and \(31 = 30 + 1\) and that \(19 = 20 - 1\) and \(21 = 20 + 1\) and recalling from problem 1c above that \((x - y)(x + y) = x^2 - y^2\). We can rewrite the expression as \((30 - 1)(30 + 1) + (20 - 1)(20 + 1) = 30^2 - 1^2 + 20^2 - 1^2 = 900 - 1 + 400 - 1 = 1300 - 2 = 1298\).

4. Let our positive integers be called \(x\) and \(y\), with \(x > y\). We know that \(x + y = 11\) and that \(x^2 - y^2 = 55\). We are trying to determine the value of \(x - y\). Once again, recall from problem 1c that \(x^2 - y^2 = (x - y)(x + y)\). Let's rewrite our second equation as \((x - y)(x + y) = 55\). Substituting 11 for \(x + y\), we get \(11(x - y) = 55\), so \(x - y = 5\).

The Problems are solved in the MATHCOUNTS Mini video.

Follow-up Problems

5. Notice that our expression is the difference of squares since \(2^{1024} - 1 = (2^{512})^2 - 1^2\). We can rewrite \((2^{512})^2 - 1^2\) as \((2^{512} - 1)(2^{512} + 1)\). The second factor is very large, but the first factor is smaller and is also the difference of squares, namely \((2^{256})^2 - 1^2\). This expression can be rewritten as \((2^{256} - 1)(2^{256} + 1)\). Again, the first factor is smaller than the second factor and can be rewritten because it is the difference of squares, namely \((2^{128})^2 - 1^2\). If we continue rewriting the expression in this manner, we eventually get to a factor of \(2^4 - 1^2 = (2^2 - 1)(2^2 + 1) = (3)(5)\). Neither of these factors can be broken down further. The only smaller prime that might be factor is 2. However, \(2^{1024} - 1\) is odd and doesn’t have a factor of 2. So, 3 and 5 are the two smallest prime factors, and their product is \(3 \times 5 = 15\).
Let our positive integers be $x$ and $y$, with $x > y$. We know that $x^2 - y^2 = 2009$, so $(x + y)(x - y) = 2009$. To maximize the value of $x - y$, let's try $y = 1$. Substituting, we get $x^2 - 1^2 = 2009$, so $x^2 = 2010$. However, $x = \sqrt{2010}$ is not an integer. The smallest perfect square that is greater than 2009 would be 2025. So, to get $x = \sqrt{2025} = 45$, we must have $x^2 = 2025 = 2009 + 16$. Therefore, $x^2 - 16 = 2009$ and $x^2 - 4^2 = 2009$. In this case, $x = 45$ and $y = 4$ and $x - y = 45 - 4 = 41$. (Note: the other $x$ and $y$ that satisfy the equation, $x = 147$ and $y = 140$, have difference $147 - 140 = 7$.)

Since $a$, $b$ and $c$ are consecutive positive odd integers, we can write $a$ and $c$ in terms of $b$ as $a = b - 2$ and $c = b + 2$. We know that $b^2 - (b - 2)^2 = 344$, and $(b + 2)^2 - b^2$ is what we are asked to find. Let's start by simplifying $b^2 - (b - 2)^2 = 344$. We have $b^2 - [(b - 2)(b - 2)] = 344 \rightarrow b^2 - (b^2 - 4b + 4) = 344 \rightarrow b^2 - b^2 + 4b - 4 = 344 \rightarrow 4b - 4 = 344 \rightarrow 4b = 348$. Next, let's simplify $(b + 2)^2 - b^2$. We get $(b + 2)(b + 2) - b^2 \rightarrow b^2 + 4b + 4 - b^2 \rightarrow 4b + 4$. Since $4b = 348$, we have $4b + 4 = 348 + 4 = 352$.

Let $m$ and $p$ represent Minnie's and Paul's ages, respectively. We know that $m + p^2 = 7308$ and $p + m^2 = 6974$. Subtracting these two equations yields $m + p^2 - p - m^2 = 334$, which can be rewritten as $-(p - m) + (p - m)(p + m) = 334$ or $(p - m)(-1 + p + m) = 334$. Since $334 = 2 \times 167$ and 167 is prime, it follows that $p - m = 2$ and $-1 + p + m = 167$. Therefore, $p + m = 168$. 

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