MATHCOUNTS® Minis **May 2019 Activity Solutions**

Warm-Up!

1. We are told that A(0, 1), B(2, 1) and C(4, 0) are three vertices of parallelogram ABCD. By properties of parallelograms, we know that sides AD and BC are parallel and congruent to each other, as are sides AB and DC. We can also see that side AB is on the line $y = 1$ and side DC is on the line $y = 0$, so the fourth vertex is D(x, 0). Since AB = 2 − 0 = 2, it follows that DC = 2. That means $x = 4 - 2 = 2$. Therefore, for vertex D(2, 0), we get 2 + 0 = **2**.

2. The lattice points in the rectangle's interior have coordinates with $1 \le x \le 6$ and $1 \le y \le 3$, for integers x and y. Thus, there are six possible x-coordinates, each of which can be paired with one of the three possible y-coordinates. So, the interior of this rectangle contains 6 × 3 = **18** lattice points.

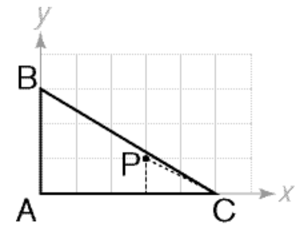
3. This triangle has vertices at the intersections of pairs of these three lines. Let's find the intersection of $y = x + 4$ and $y = 0$. Substituting 0 for y in the first equation, we get $0 = x + 4$ and $x = -4$. Therefore, these lines intersect at (−4, 0). Next, we'll find the intersection of $y = x + 4$ and $x + 3y = 12$. Substituting $x + 4$ for y the second equation, we get $x + 3(x + 4) = 12$. Simplifying, we see that $x + 3x + 12 = 12 \rightarrow 4x = 0 \rightarrow x = 0$ and $y = 0 + 4 = 4$. So, these lines intersect at (0, 4). Finally, we'll determine the intersection of $y = 0$ and $x + 3y = 12$. Substituting 0 for y in the first equation, we get $x = 12$. Therefore, these two lines intersect at (12, 0). This triangle has base length 12 − (−4) = 16, height 4 − 0 = 4 and area (1/2)(16)(4) = **32** units^2

4. The segment with endpoints (−2, 3) and (6, −7) has midpoint ((−2 + 6)/2, (3 − 7)/2) = (2, −2). This point is equidistant from the endpoints of the segment, as are all points on a line through this point and perpendicular to this segment. The slope of the segment is (−7 − 3)/(6 + 2) = −10/8 = −5/4. That means the slope of the line perpendicular to the segment through (2, −2) is 4/5. Substituting m, x and y into $y - y_1 = m(x - x_1)$, we have $y + 2 = (4/5)(x - 2) \rightarrow y + 2 = (4/5)x - 8/5 \rightarrow (-4/5)x + y = -18/5$. Again, we need a leading coefficient of 1, so we multiply by −5/4 to get $x - (5/4)y = 9/2$. Thus, B = **−5/4**.

The Problems are solved in the MATHCOUNTS® Mini video.

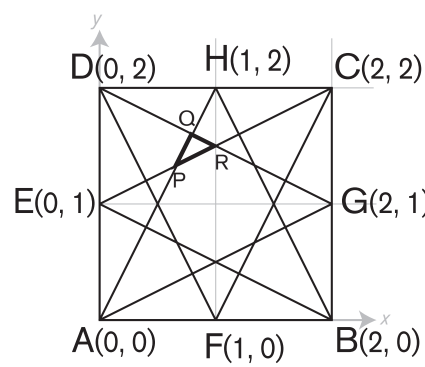
Follow-up Problems

5. Pentagon ABCDE is composed of a square with side length 4 units and an isosceles triangle with a base of 4 units and height h units. Adding the area of the square and the area of the triangle gives us the area of the entire figure, which we are told is 40 units^2. So, $4^2 + (1/2)(4)h = 40 \rightarrow 16 + 2h = 40 \rightarrow 2h = 24 \rightarrow h = 12$. It follows, then, that the third vertex of the triangle is C(2, 16). Side BC has slope (16 − 4)/(2 − 0) = 12/2 = 6, side BC intersects the point (0 + 1, 4 + 6) = (1, 10). By symmetry, side CD intersects the point (3, 10). Notice that lattice points in the interior of pentagon ABCDE all have an x-coordinate of 1, 2 or 3. When $x = 1$, there are interior lattice points when $0 < y < 10$, for integer y, for a total of 9 lattice points. By symmetry, the same is true when $x = 3$, for another 9 lattice points. When $x = 2$, there are lattice points when $0 < y < 16$, for integer y, for a total of 15 lattice points. Therefore, the total number of lattice points in the interior of pentagon ABCDE is 9 + 9 + 15 = **33** lattice points.



6. We know that the distances from P to vertices A and B are $\sqrt{10}$ units and $\sqrt{13}$ units, respectively. If we let the coordinates of point P be (x, y), then $\sqrt{[(x - 0)^2 + (y - 0)^2]} = \sqrt{10} \rightarrow x^2 + y^2 = 10$ and $\sqrt{[(x - 0)^2 + (y - 3)^2]} = \sqrt{13} \rightarrow x^2 + (y - 3)^2 = 13 \rightarrow x^2 + y^2 - 6y + 9 = 13 \rightarrow x^2 + y^2 - 6y = 4$. Subtracting the equations $x^2 + y^2 = 10$ and $x^2 + y^2 - 6y = 4$, we get $6y = 6 \rightarrow y = 1$. Substituting for y in the equation $x^2 + y^2 = 10$ yields $x^2 + 1 = 10 \rightarrow x^2 = 9 \rightarrow x = 3$. Now we know that the coordinates of P are (3, 1). Notice that segment PC is the hypotenuse of a triangle with leg lengths 1 and 2. Using the Pythagorean Theorem, we have $1^2 + 2^2 = PC^2 \rightarrow 5 = PC^2 \rightarrow PC = \sqrt{5}$. So the distance from P to vertex C is $\mathbf{\sqrt{5}}$ units.

7. We solved the previous problems using the properties of a segment bisector, but for this problem, we will use the properties of an *angle bisector*. Recall that an angle bisector is the ray originating at the vertex of an angle that divides the angle into two congruent angles. By theorem, every point on the line that bisects a given angle is equidistant from the sides of the angle. We are told that the point (x, $2x$), which is on the line $y = 2x$, is equidistant from both sides of the angle formed by segments AB and BC. It follows, then that the (x, $2x$) must be a point on the line that bisect $\angle$ABC. The slope of side AB is $(3 - 0)/(5 - 2) = 3/3 = 1$, and the slope of side BC is $(4 - 0)/(-2 - 2) = 4/(-4) = -1$. So, side AB is perpendicular to side BC and $m\angle ABC = 90°$. That means that the bisector of $\angle$ABC is the vertical line through (2, 0) given by $x = 2$. Thus, $x = \mathbf{2}$.



8. The figure shows square ABCD on the coordinate grid with vertices A(0, 0), B(2, 0), C(2, 2) and D(0, 2), and the midpoints of the sides are E(0, 1), F(1, 0), G(2, 1) and H(1, 2). The sides of triangle PQR are on segments AH, EC and DG. We'll find the linear equation for these three segments, then determine the points of intersection for pairs of these segments to get the coordinates of P, Q and R. Segment AH has slope $(2 - 0)/(1 - 0) = 2$ and y-intercept 0. Segment AH is given by $y = 2x$. Next, segment EC has slope $(1 - 2)/(0 - 2) = 1/2$ and y-intercept 1. Segment EC is given by $y = (1/2)x + 1$. Finally, segment DG has slope $(1 - 2)/(2 - 0) = -1/2$ and y-intercept 2. Segment DG is given by $y = (-1/2)x + 2$. Segments AH and EC intersect at P. Setting their equations equal to each other, we get $2x = (1/2)x + 1 \rightarrow (3/2)x = 1 \rightarrow x = 2/3$ and $y = 2(2/3) = 4/3$. So, one vertex of triangle PQR is P(2/3, 4/3). Segments AH and DG intersect at Q. Setting their equations equal to each other, we get $2x = (-1/2)x + 2 \rightarrow (5/2)x = 2 \rightarrow x = 4/5$ and $y = 2(4/5) = 8/5$. So, another vertex of triangle PQR is Q(2/3, 4/3). Segments EC and DG intersect at R. Setting their equations equal to each other, we get $(1/2)x + 1 = (-1/2)x + 2 \rightarrow x + 1 = 2 \rightarrow x = 1$ and $y = (1/2)(1) + 1 = 3/2$. So, the third vertex of triangle PQR is R(1, 3/2). Finally, to determine the value of the ratio QR/PQ, we will determine the lengths of sides PQ and QR using the distance formula. We have $QR = \sqrt{[(4/5 - 1)^2 + (8/5 - 3/2)]} = \sqrt{[(1/5)^2 + (1/10)^2]} = \sqrt{(1/25 + 1/100)} = \sqrt{(5/100)} = \sqrt{5}/10$ units. We also get $PQ = \sqrt{[(2/3 - 4/5)^2 + (4/3 - 8/5)]} = \sqrt{[(-2/15)^2 + (-4/15)^2} = \sqrt{(4/225 + 16/225)} = \sqrt{(20/225)} = 2\sqrt{5}/15$ units. Therefore, the ratio $QR/PQ = (\sqrt{5}/10)/(2\sqrt{5}/15) = (\sqrt{5}/10)(15/(2\sqrt{5})) = (1/2)(3/2) = \mathbf{3/4}$.