MATHCOUNTS® Minis **April 2019 Activity Solutions**

## Warm-Up!

1. A number is divisible by 4 if the two-digit number formed by the tens and ones digits is a multiple of 4. Since we are looking for the greatest four-digit number, we should try to put the 7 and 5 in the thousands and hundreds places, respectively. Doing so leaves the 1 and 2 for the remaining places in the four-digit number. We are looking for the greatest number, so we first consider the number 7521, but the two-digit number 21 is not a multiple of 4. The two-digit number 12, however, is a multiple of 4, which leads to our answer, **7512**.

2. We are looking for the units digit when we multiply all the odd numbers from 1 to 2015, inclusive. If we multiply $1 \times 3 \times 5 \times 7 \times 9$, and look only at the units digit, we see this product has a units digit of 5. Because the same digits are multiplied, the product of the odd numbers from 11 to 19 will have a units digit of 5, as will the product of the odd numbers from 21 to 29 and from 31 to 39, and so on. Therefore we know $1 \times 3 \times 5 \times \cdots \times 2009$ will have a units digit of 5, since 5 multiplied by itself, any number of times, will result in a units digit of 5. The only numbers left to multiply, then, are 2011, 2013 and 2015. Multiplying these three numbers will give us the same units digit as multiplying $1 \times 3 \times 5$, which is 5. The units digit for the product $1 \times 3 \times 5 \times \cdots \times 2015$, therefore, also will be **5**.

3. To evaluate $5/3 \times 6/4 \times 7/5 \times 8/6 \times \cdots \times 120/118$, we can start by cancelling common factors in the numerator and denominator. Doing so, results in the much simpler expression $(119 \times 120)/(3 \times 4) = 119 \times 10 =$ **1190**.

4. Since we want to maximize the difference, we should try the difference of a number in the nine-hundreds and a number in the one-hundreds.The greatest difference is $921 - 129 = 931 - 139 = 941 - 149 = 951 - 159 = 961 - 169 = 971 - 179 = 981 - 189 =$ **792**.

**The Problems** are solved in the MATHCOUNTS® Mini video.

## Follow-up Problems

5. In order for this four-digit number to be divisible by 9, the sum of the digits must be a multiple of 9. We are looking for the smallest four-digit number that is divisible by 9 and that has two even digits and two odd digits. The smallest possbile four-digit number is 1000. Let's start with 10_ _. So far, we have $1 + 0 = 1$. We won't be able to find a pair of digits, one even and one odd, whose sum is 8 because and even plus and odd is always odd. So, let's try to find a pair of digits, one even and one odd, whose sum is 17. The only pair that works is 8 and 9. Thus, the smallest four-digit number that works is **1089**.

6. Consider the four-digit number ABCD, where each letter represents a digit. We have $A + B = C$, $B + C = D$ and $C + D = 10A + B$. If we rewrite the last equation as $D = 10A + B - C$, then we can substitute this for D in the second equation. We get $B + C = 10A + B - C \rightarrow 2C = 10A \rightarrow C = 5A$. Since 1 is the only nonzero one-digit number that yields another one-digit number when multiplied by 5, it follows that $A = 1$ and $C = 5$. So $1 + B = 5$ and $B = 4$. Also, we have $4 + 5 = D$, so $D = 9$. Thus, the four-digit number is **1459**.

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7. If $2015 + a = b$, then $a = b - 2015$. Since $a$ and $b$ are both positive integers, we start subtracting 2015 from each of the possible $b$ palindromes, beginning with 2112, which is the smallest palindrome greater than 2015. As the table shows, the first difference we obtain that also is a palindrome is 2772 – 2015 = **757**.

| ***b*** | 2112 | 2222 | 2332 | 2442 | 2552 | 2662 | **2772** |
|---|---|---|---|---|---|---|---|
| ***a*** | 97 | 207 | 317 | 427 | 537 | 647 | **757** |

8. Let's, first, try a few different combinations of positive odd integers $x$, $y$ and $z$ to see what the remainder is when the sum of their squares is divided by 4. The table below shows four such scenarios.

| $x$ | $y$ | $z$ | $x^2 + y^2 + z^2$ | $\frac{x^2 + y^2 + z^2}{4}$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 3 | 0 R.3 |
| 1 | 1 | 3 | 11 | 2 R.3 |
| 1 | 3 | 3 | 19 | 4 R.3 |
| 1 | 3 | 5 | 35 | 8 R.3 |

It appears that for all positive odd integers $x$, $y$ and $z$, when the sum of their squares is divided by 4, the remainder will always be 3. Let's see if we can prove this algebraically. For nonnegative integers $u$, $v$ and $w$, let $x = 2u + 1$, $y = 2v + 1$ and $z = 2w + 1$, since $x$, $y$ and $z$ are odd integers. Then the sum of the squares of $x$, $y$ and $z$ is $x^2 + y^2 + z^2 = (2u + 1)^2 + (2v + 1)^2 + (2w + 1)^2 = 4u^2 + 4u + 1 + 4v^2 + 4v + 1 + 4w^2 + 4w + 1 = 4(u^2 + u + v^2 + v + w^2 + w) + 3$. This proves that the sum of the squares of all positive odd integers $x$, $y$ and $z$ will always leave a remainder of **3** when divided by 4.

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