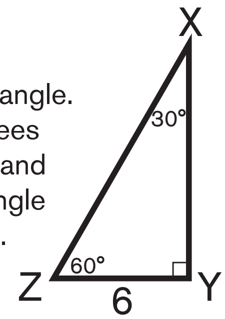
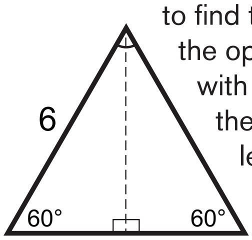
MATHCOUNTS® Minis

# March 2019 Activity Solutions

## Warm-Up!

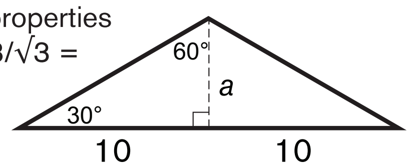


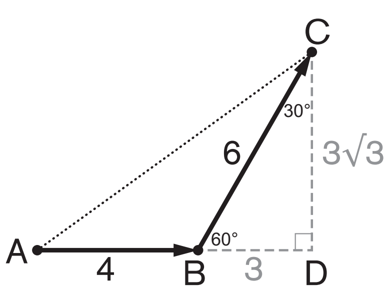

1. Since the $m\angle X$ = 30 degrees and $m\angle Y$ = 90 degrees, it follows that the $m\angle Z$ = 180 – ( 90 + 30) = 180 – 120 = 60 degrees. Therefore, △XYZ is a 30-60-90 right triangle. In a 30-60-90 right triangle, the length of the longer leg, which is opposite the 60-degrees angle, is $\sqrt{3}$ times the length of the shorter leg, which is opposite the 30-degree angle, and the length of the hypotenuse is twice that of the shorter leg. As the figure shows, in triangle XYZ, side YZ is the shorter leg, side XY is the longer leg and side XZ is the hypotenuse. We are told YZ = 6 units, so it follows that XY = **$6\sqrt{3}$** units and XZ = **12** units.




2. We know the base of this equilateral triangle has length 6 units. To determine the area we need to find the height of the triangle. If we draw the altitude from the vertex angle of the triangle to the opposite side, as shown, it divides the triangle into two 30-60-90 right triangles, each with a hypotenuse of length 6 units. By properties of 30-60-90 right triangles, the length of the shorter leg of each of the right triangles is (1/2) × 6 = 3 units. The length of the longer leg, which is the height of the equilateral triangle, then is $3\sqrt{3}$ units. We now see that the area of this equilateral triangle is $(1/2)(6)(3\sqrt{3})$ = **$9\sqrt{3}$** units$^2$.

3. As the figure shows, if we draw the altitude from the vertex angle to the base, we create two congruent 30-60-90 right triangles. Since the isosceles triangle has base length 20 feet, the long leg of each of these right triangles has length 10 feet. Let *a* represent the altitude of the isosceles triangle, which is also the length of the short let of the right triangle. By properties of 30-60-90 right triangles, we know that $a\sqrt{3} = 10$, so $a = 10/\sqrt{3} \times \sqrt{3}/\sqrt{3} = 10\sqrt{3}/3$ feet. So, the isosceles triangle has base length 20 feet, height $10\sqrt{3}/3$ feet and area $1/2 \times 20 \times 10\sqrt{3}/3$ = **$100\sqrt{3}/3$** ft$^2$.

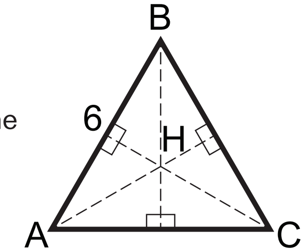


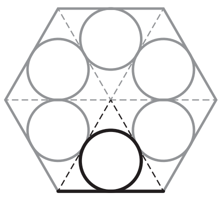

4. Let's sketch out Billy's route as described. From location A, he travels 4 miles east to location B. He then makes a 60-degree turn northward and then travels 6 miles to location C. In the figure, the dotted segment drawn from A to C creates △ABC. We are asked to determine how far Billy is from his starting point, or AC in our figure. If we extend side AB to a point D that is the intersection of a perpendicular segment drawn from point C, we create right triangles BCD and ACD. Angle CBD has measure 60 degrees, and we have constructed angle BDC to be a right angle. Therefore, △BCD is a 30-60-90 right triangle with hypotenuse of length 6 miles. By properties of 30-60-90 right triangles, we determine that BD = (1/2)(6) = 3 miles, and CD = $3\sqrt{3}$ miles. We now know that right triangle ACD, with hypotenuse AC, has legs of lengths 4 + 3 = 7 miles and $3\sqrt{3}$ miles. We can now use the Pythagorean Theorem to find AC as follows: $AC = \sqrt{(7^2 + (3\sqrt{3})^2)} = \sqrt{(49 + 27)} = \sqrt{76} = 2\sqrt{19}$. Thus, Billy's distance from his starting point is **$2\sqrt{19}$** miles.

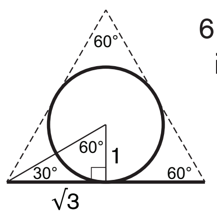
**The Problems** are solved in the MATHCOUNTS® Mini video.

## Follow-up Problems

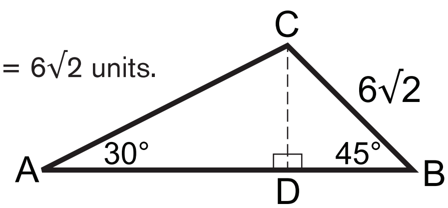



5. Recall from the solution of the third problem in the video that the altitudes of an equilateral triangle, which bisect the angles of the triangle, intersect at a point in the center of the triangle that is equidistant from each side. Drawing these altitudes in $\triangle ABC$, as shown, we see that six congruent 30-60-90 right triangles are created. We are asked to find AH, to which we will assign the variable *x*. Since side AH is the hypotenuse of one of these 30-60-90 right triangles, is follows that the shorter leg has length $x/2$, and the longer leg has length $(x/2)\sqrt{3}$. But we know that the length of each side of $\triangle ABC$ is 6, and the length of the longer leg of one of the 30-60-90 right triangles is half that amount, $(1/2) \times 6 = 3$. So, we have $(x/2)\sqrt{3} = 3 \rightarrow x/2 = 3/\sqrt{3} \rightarrow x\sqrt{3} = 6 \rightarrow x = 6/\sqrt{3}$. Simplifying, we see that $x$ = AH = $\mathbf{2\sqrt{3}}$ units.






6. As the figure on the left shows, a regular hexagon can be divided into six congruent equilateral triangles. For this particular regular hexagon, notice that each of the six circles is inscribed in one of these equilateral triangles. We will focus on the triangle and inscribed circle in the center at the bottom of that figure. The figure on the right shows that triangle and inscribed circle enlarged. Let's first draw the radius of the circle that is perpendicular to the base of the triangle. Then we draw a segment from the center of the circle to a vertex, as shown. We have created a 30-60-90 right triangle, where the short leg is a radius of the circle, which we know has length 1 unit. Using properites of 30-60-90 right triangles, we determine that the long leg of this triangle has length $\sqrt{3}$ units. Notice that the long leg of this triangle is half the side length of the equilateral triangle. Therefore, the equilateral triangle has side length $2\sqrt{3}$ units. Since the base of the equilateral triangle is a side of the hexagon, it follows that the hexagon has side length $2\sqrt{3}$ units. The perimeter of the hexagon, then, is $6 \times 2\sqrt{3}$ = $\mathbf{12\sqrt{3}}$ units.




7. Triangle ABC has $m\angle A = 30$ degrees, $m\angle B = 45$ degrees and BC = $6\sqrt{2}$ units. A perpendicular segment is drawn from vertex C to point D on side AB, as shown. Since $m\angle BDC = 90$ degrees, it follows that $m\angle BCD = 45$ degrees, making triangle BCD an isosceles right triangle. If BD = *x* units, then we have $x^2 + x^2 = (6\sqrt{2})^2 \rightarrow 2x^2 = 72 \rightarrow x^2 = 36 \rightarrow x = 6$ units. So, BD = CD = 6 units. For triangle ADC, since $m\angle A = 30$ degrees and $m\angle ADC = 90$ degrees, then $m\angle DCA = 60$ degrees. Using properties of 30-60-90 right triangles, we have AD = $6\sqrt{3}$ units and AC = 12 units. So, AB = AD + BD = $6\sqrt{3} + 6$ units and AB + AC = $6\sqrt{3} + 6 + 12 \approx$ **28.4** units.

8. Let's extend segments AD and BC until they intersect at point E, as shown. Notice that $m\angle EBA = 180 - 120 = 60$ degrees, and $m\angle BAE = 180 - 90 = 90$ degrees. That means the $m\angle E = 30$ degrees, and $\triangle ABE$ is a 30-60-90 right triangle. We know that AB = 3, so using the properties of 30-60-90 right triangles, we see that EB = 2 × 3 = 6. Now consider right triangle CDE with $m\angle C = 90$ degrees and $m\angle E = 30$ degrees. It follows that $m\angle D = 60$ degrees making $\triangle CDE$ a 30-60-90 right triangle. The length of the longer leg is EC = EB + BC = 6 + 4 = 10. Segment CD is the shorter leg of $\triangle CDE$. Therefore, according to the properties of 30-60-90 right triangles, we have CD = EC/$\sqrt{3}$ = 10/$\sqrt{3}$ = $\mathbf{10\sqrt{3}/3}$ units.