MATHCOUNTS® Minis # January 2019 Activity Solutions

Warm-Up!

1. If a © $b = a^2 + ab + b^2$, then (3 © 4) − 12 = $(3^2 + 3 \cdot 4 + 4^2) - 12 = (9 + 12 + 16) - 12 =$ **25**.

2. If $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, then $f(g(3)) = (2 \cdot 3 + 1)^2 - 3 = 7^2 - 3 = 49 - 3 =$ **46**.

3. It's probably safe to assume that D doesn't represent 0, 1, 2, 5, 7 or 8 since those digits are shown in the equation. That leaves 3, 4, 6 and 9 as the possible digit that D represents. Since there are only four possibilities, we could start trying them to see which one makes the equation true. If D represents 3, we get 23 × 351 = 8073. If D represents 4, 6 or 9, the equation is not true. Therefore, D must represent the digit **3**.

4. The rule defining this sequence tells us that any term is the sum of the previous term and two times the term two prior. We can follow this rule using the three known terms to find additional terms. The sequence begins as 1, 1, 3. Using the rule we know, $A_4 = 3 + 2(1) = 5$, $A_5 = 5 + 2(3) = 11$ and finally, $A_6 = 11 + 2(5) =$ **21**.

The Problem is solved in the **MATHCOUNTS® Mini** video.

Follow-up Problems

5. Comparing 5 & 3 = 18 and 10 & 3 = 72, we see that the result of (2 × 5) & 3 is 72 ÷ 18 = 4 times the result of 5 & 3. This suggests that the first number might get squared, which means that $m = 2$. Next, comparing 5 & 3 and 5 & 6, we see that the result of 5 & (2 × 3) is 36 ÷ 18 = 2 times as great as 5 & 3. This suggests that the second number might be raised to the first power, which means that $n = 1$. Substituting 2 for m and 1 for n in the expression $k \times A^m \times B^n$ for $A = 5$ and $B = 3$ yields $k \times 5^2 \times 3^1 = 18 \rightarrow 75k = 18 \rightarrow k = 18/75 = 6/25$. Let's confirm this with 10 & 3 and 5 & 6. We have $k \times 10^2 \times 3^1 = 72 \rightarrow 300k = 72 \rightarrow k = 72/300 = 6/25$ and $k \times 5^2 \times 6^1 = 36 \rightarrow 150k = 36 \rightarrow k = 36/150 = 6/25$. Now that we've confirmed that $k = 6/25$, we can calculate the value of 10 & 6 = $(6/25) \times 10^2 \times 6^1 = 3600/25 =$ **144**.

6. Look at the first term in parentheses and the second term in parentheses, notice that both begin with 1 ⊕ 2 ⊕ 3 ⊕ 4 ⊕ 5 ⊕ 6 ⊕ 7. Let's define x = 1 ⊕ 2 ⊕ 3 ⊕ 4 ⊕ 5 ⊕ 6 ⊕ 7. Now look at the first term in parentheses and compute in terms of x:

$$x \oplus 9 = \sqrt{x^2 + 9^2}$$

$$\sqrt{x^2 + 9^2} \oplus 10 = \sqrt{x^2 + 9^2 + 10^2}$$

Similarly, the second term in parentheses can be expressed as:

$$x \oplus 8 = \sqrt{x^2 + 9^2}$$

$$\sqrt{x^2 + 8^2} \oplus 9 = \sqrt{x^2 + 8^2 + 9^2}$$

Plugging these back into the original expression and calculating the final value we get:

$$\sqrt{x^2 + 9^2 + 10^2} \ominus \sqrt{x^2 + 8^2 + 9^2} = \sqrt{x^2 + 9^2 + 10^2 - (x^2 + 8^2 + 9^2)} = \sqrt{10^2 - 8^2} = \sqrt{36} = \mathbf{6}$$

7. The nth term of a sequence is $a_n = (-1)^{n+1}(3n + 2)$. Solving for the value of the first four terms:

$a_1 = (-1)^2(3 + 2) = 5$

$a_2 = (-1)^3(6 + 2) = -8$

$a_3 = (-1)^4(9 + 2) = 11$

$a_4 = (-1)^5(12 + 2) = -14$

A pattern is emerging: $a_1 + a_2 = 5 + (-8) = -3$ and $a_3 + a_4 = 11 + (-14) = -3$. Each two terms of the series will sum to −3. There are 100 terms in the series so the total sum is 50 × (−3) = **−150**.

8. Let's start with 1:

$f(1) = 1^2 + 1 = 1 + 1 = 2$

$f(f(1)) = f(2) = 2/2 = 1$.

At this point, we're going to repeat 2, 1, 2, 1, etc. Now start with 2:

$f(2) = 2/2 = 1$

$f(f(2)) = f(1) = 2$

And again, we'll just repeat 1, 2. Now start with 3.

$f(3) = 3^2 + 1 = 9 + 1 = 10$

$f(f(3)) = f(10) = 10/2 = 5$

$f(f(f(3))) = f(5) = 5^2 + 1 = 26$

$f(f(f(f(3)))) = f(26) = 13$

Continued application of the function f will only result in larger and larger values. Therefore, 3 is not a part of this (and none of the rest of the odd values). Now start with 4.

$f(4) = 4/2 = 2$

$f(f(4)) = f(2) = 2/2 = 1$

So 4 works. We already know that 5 will not so let's move to 6.

$f(6) = 6/2 = 3$

$f(f(6)) = f(3) = 10$

We now have enough information. For $n = 1, 2, 4$, we can have the situation where multiple applications of f will result in the value of 1. Odd values starting with 3 will never result in the value of the function being 1 because the value just gets bigger and bigger. That leaves even numbers. We see that 4 works but 6 doesn't. This is because even numbers when f is continuously applied continue to be halved. If the even number has an odd factor, then f will eventually be applied to an odd number and will never reach 1. If the even number is a power of 2, then continually application of f will eventually arrive at 1. The numbers 1, 2, 4, 8, 16, 32 and 64 will satisfy the conditions. Meaning there are **7** integers.